Optimal Credit Fluctuations*

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Abstract

Under which conditions are extrinsic credit fluctuations socially optimal? In order to answer this question we characterize constrained-efficient allocations in an infinite horizon, two-good economy with limited commitment for two market structures, random pairwise meetings and centralized meetings. If agents meet bilaterally, then constrained-efficient allocations specify the highest stationary output level that is incentive feasible, and it is implemented with take-it-or-leave-it offers and "not-too-tight" solvency constraints. If agents meet in a centralized location, constrained-efficient allocations can be non-stationary, in which case they feature a credit boom followed by stagnation due to "too-tight" solvency constraints. We also characterize constrained-efficient allocations under partial commitment by the planner. If commitment is low, the economy experiences rare but pronounced credit crunches. If commitment is high, the economy experiences infrequent but large credit booms.

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1 Introduction

Most financial crises are preceded by a large build-up in private agents' debt followed by a credit contraction and a prolonged period of stagnation with low economic activity.[1] These fluctuations in the use and availability of credit are typically viewed as socially inefficient as they create excess volatility relative to economic fundamentals. In this paper we challenge this view and provide a simple environment where endogenous credit fluctuations can occur in equilibrium and are socially desirable ex ante.

The environment we consider is an infinite-horizon, two-good economy similar to the one in Gu et al. (2013b), GMMW thereafter. In the presence of idiosyncratic shocks, this economy features a role for intertemporal trades interpreted as unsecured credit arrangements (e.g. consumption of one good in exchange for promises of another good). Since agents lack commitment, those trades are sustained through public monitoring.[2] Using an equilibrium approach, GMMW show that this economy can generate credit cycles due to a pecuniary externality according to which endogenous debt limits affect the relative price of goods (hence, the importance of having two goods). In brief, high debt limits in the future generate high (relative) prices of goods consumed on credit that reduce the private gains from having access to credit in the future. As a result, the punishment from being excluded from credit is reduced, which generates a low debt limit in the current period.

Debt limits in GMMW are obtained by imposing the "not-too-tight" solvency constraints of Alvarez and Jermann (2000), AJ thereafter, according to which in every period agents can issue the maximum amount of debt that is incentive-compatible with no default. While these constraints implement constrained-efficient allocations in the AJ one-good economy, it is not necessarily the case in an economy with multiple goods, as pointed out by Kehoe and Levine (1993). Therefore, instead of taking as primitives arbitrary solvency constraints, we derive them from a mechanism design approach which characterizes incentive-feasible allocations that maximize ex-ante welfare (see Wallace 2010 for a survey of this approach in monetary economics). In contrast to most of this literature, with the noticeable exception of Cavalcanti and Erosa (2008), we do not restrict the set of allocations to stationary ones and characterize instead the dynamic contracting problem between the planner and private agents. In doing so, we distinguish two market structures commonly used in the literature, pairwise meetings and large-group meetings, and we impose the corresponding core requirement.

[1] Rogoff (2016) argue for the existence of debt supercycles characterized by credit booms and credit crunches. Lo and Rogoff (2015) argue that sluggish economic growth after the onset of the financial crisis is due to significant pockets of private, external and public debt overhang.

[2] In the absence of public record keeping, the environment corresponds to the New-Monetarist framework of Lagos and Wright (2005) with a few differences.

We establish the following results. If agents are matched bilaterally then the constrained-efficient allocation corresponds to the incentive-feasible, stationary allocation with the highest output level. It is implemented with take-it-or-leave-it offers by borrowers and "not-too-tight" solvency constraints, which extends the AJ welfare theorem to economies with pairwise meetings. In such an economy, credit fluctuations would lower welfare, thereby capturing the common wisdom.

If all agents meet together in a centralized location, then the core requirement is equivalent to competitive pricing. Under strictly convex production costs, the price of credit goods increases with aggregate consumption and hence buyers' surpluses vary in a non-monotone fashion with aggregate borrowing. In turn, large borrowing is incentive-compatible if buyers can anticipate large surpluses in the future. Therefore, in order to extract large social gains from trade in the current period, the planner must promise low prices for future consumption, which requires lower aggregate consumption in the future. In other words, the planner faces a trade-off between contemporaneous and future output. If preferences are such that the temptation to renege on future promises is low (in a sense to be made precise below), then the constrained-efficient allocation is non-stationary. The initial period features an output level that is larger than the highest steady state, which we interpret as a credit boom. It is followed by a long-lasting stagnation where output is lower than the highest steady state due to solvency constraints that are overly tight (tighter than what is required to make repayment incentive-feasible).

The non-stationary constrained-efficient allocation is time inconsistent in that the planner would like to revise the allocation after the initial credit boom. Our result is robust to this time-inconsistency in the following sense. We introduce partial commitment according to which the planner can reoptimize infrequently when a sunspot state is realized. If the planner's commitment power is low, the economy experiences rare but pronounced credit crunches. If it is high, then the economy experiences infrequent but large credit booms.

1.1 Related literature

Seminal contributions on limited-commitment economies include Kehoe and Levine (1993), Kocherlakota (1996), and AJ. We differ from Kocherlakota (1996) and AJ in that we study a two-good production economy under alternative market arrangements. Our environment is a variant of the Lagos-Wright (2005) and Rocheteau-Wright (2005) frameworks, in that we use a two-stage structure and quasi-linear preferences, but we replace currency with a public record-keeping technology, as in Sanches and Williamson (2010, Section 4). Following GMMW, preferences are generalized to parametrize incentives to renege on obligations that span across multiple stages. Mechanism design was first applied to this environment with currency by Hu,

Kennan, and Wallace (2009).[3] Our implementation results are related to the Second Welfare Theorem in AJ in that we provide a necessary and sufficient condition under which this theorem applies to our environment. Kehoe and Levine (1993, Section 7) conjectured that punishments based on partial exclusion might allow the implementation of socially desirable allocations.[4] This conjecture is verified in our economy with the caveat that the extent of exclusion has to vary over time. Gu et al. (2013a, Section 7) studies optimal dynamic contracts between a lender and a borrower in a similar environment with pairwise meetings. In contrast, we characterize the constrained-efficient allocation that maximizes ex-ante social welfare, and we consider both pairwise meetings and centralized meetings. Under centralized meetings the planner internalizes the effects of aggregate consumption on relative prices, which is key for the existence of non-stationary constrained-efficient allocations. Finally, we are related to the vast literature on equilibrium credit cycles and pecuniary externalities including Kiyotaki and Moore (1997) and Myerson (2012), among others.

2 Environment

Time is discrete, goes on forever, and starts with period 0. Each date has two stages, 1 and 2. There is a single, perishable good at each stage. There is a continuum of agents of measure two divided evenly into buyers and sellers.[5] The labels "buyer" and "seller" refer to agents' roles in stage 1: only the sellers can produce the stage-1 good and only the buyers consume it. In each period a fraction $\alpha \in (0, 1]$ of buyers and sellers chosen at random in the whole population are matched together. The remaining $1 - \alpha$ are unmatched and stay in autarky for one period. We will make two assumptions regarding the matching process: either matched agents are allocated in pairs composed of one buyer and one seller or they meet in a single large group. Matches are destroyed at the end of each period.

Preferences are additively separable over dates and stages. The stage-1 utility of a seller who produces $y \in \mathbb{R}_+$ is $-v(y)$, while that of a buyer who consumes y is $u(y)$, where $v(0) = u(0) = 0$, v and u are strictly increasing and differentiable with v convex and u strictly concave, and $u'(0) = +\infty > v'(0) = 0$. Moreover, there exists $\tilde{y} > 0$ such that $v(\tilde{y}) = u(\tilde{y})$. We define $y^* = \arg\max [u(y) - v(y)] > 0$.

A buyer produces the stage-2 good by transforming ℓ units of stage-1 labor into ℓ units of stage-2 good.

[3] Applications of mechanism design to monetary theory include Kocherlakota (1998) and Kocherlakota and Wallace (1998), Cavalcanti and Erosa (2008), Cavalcanti and Nosal (2011), Cavalcanti and Wallace (1999), and Hu and Rocheteau (2013, 2015), among others. Wallace (2010) provides a review of the literature.

[4] Azariadis and Kass (2013) relaxed the assumption of permanent autarky and assumed that agents are only temporarily excluded from credit markets. Gu et al. (2013a) and GMMW allow for partial monitoring, which is formally equivalent to partial exclusion, except that the parameter governing the monitoring intensity, π, is time-invariant. Kocherlakota and Wallace (1998) consider the case of an imperfect record-keeping technology where the public record of individual transactions is updated after a probabilistic lag.

[5] The assumption of ex-ante heterogeneity among agents is borrowed from Rocheteau and Wright (2005). Alternatively, one could assume that an agent's role in the DM is determined at random in every period without affecting any of our results.

Production materializes in stage 2 but ℓ is perfectly observable in stage 1. Sellers do not have the technology to produce the stage-2 good. A seller's utility from consuming c units of stage-2 good is c. A buyer's utility from consuming his own output is λc where $\lambda \in [0,1]$ will play a key role to parameterize the limited commitment problem. The inability of agents to commit to future actions has some bite when $\lambda > 0$ because in stage 2 buyers have incentives to renege on promises made in stage 1 by consuming their own output. Agents' common discount factor across periods is $\beta = 1/(1+r) \in (0,1)$.

Finally, there is a monitoring technology allowing all actions of matched agents to be publicly recorded. However, agents who are matched at the beginning of a period can drop out without their participation decision being recorded.

3 Constrained-efficient allocations

We characterize constrained-efficient allocations defined as allocations that maximize social welfare subject to incentive feasibility constraints. Incentive constraints require that there are no opportunities to defect from a proposed allocation either individually or by forming coalitions, where feasible coalitions depend on the market structure. We will consider two alternative market structures, pairwise meetings and large-group meetings, that imply different core requirements.

3.1 Optimal mechanism with pairwise meetings

Suppose first that buyers and sellers are matched bilaterally. The planner chooses symmetric allocations, $\{(y_t, \ell_t)\}_{t=0}^{+\infty}$, in order to maximize the discounted sum of all match surpluses subject to incentive-feasibility conditions, where y_t is stage-1 output produced by sellers in a match and ℓ_t is the production of stage-2 goods by buyers in a match. This problem can be written as:

$$\max_{\{(y_t,\ell_t)\}} \sum_{t=0}^{+\infty} \beta^t \alpha \left[u(y_t) - \upsilon(y_t)\right] \tag{1}$$

$$\text{s.t.} \quad \lambda \ell_t \leq \sum_{s=1}^{+\infty} \beta^s \alpha \left[u(y_{t+s}) - \ell_{t+s}\right] \tag{2}$$

$$\upsilon(y_t) \leq \ell_t \leq u(y_t). \tag{3}$$

The planner's objective in (1) is the sum of buyers' and sellers' utilities in stage 1. It does not include utility flows associated with the production/consumption of the stage-2 good since having buyers produce the stage-2 good for their own consumption is socially inefficient and does not help with incentive constraints, and sellers' utility of consumption cancels out with buyers' disutility of production. Inequality (2) captures the limited commitment problem in stage 2. It guarantees that buyers prefer to deliver the stage-2 good to

sellers instead of reneging and consuming it. The left side is a buyer's utility of consuming ℓ_t units of his own production while the right side is the discounted sum of utility flows from the proposed allocation. Implicit in that formulation is the fact that in the presence of perfect monitoring the buyer can be punished by permanent autarky from departing from the proposed allocation, which generates a continuation value equal to 0. (It is easy to check that strategies that consist of punishing defectors form a sequential equilibrium. See Bethune, Hu, and Rocheteau, 2017, for a detailed characterization of all equilibria.) The conditions in (3) are participation constraints of matched buyers and sellers in stage 1. Since matched agents can choose to drop out and stay in autarky for one period without their action being recorded, we need to make sure that they receive a positive surplus from their intra-period trades, where a buyer's surplus is $u(y_t) - \ell_t$ and a seller's surplus is $-\upsilon(y_t) + \ell_t$. Coalition-proofness in pairwise meetings requires that $y_t \leq y^*$, which is satisfied endogenously (and hence ignored thereafter). We call a solution to (1)-(3) a constrained-efficient allocation (cea thereafter).

For a given $\{y_t\}$, a necessary condition for participation constraints to hold is that they hold when $\ell_t = \upsilon(y_t)$. As a result, the incentive constraints can be collapsed into a single inequality,

$$\lambda \upsilon(y_t) \leq \sum_{s=1}^{+\infty} \beta^s \alpha \left[u(y_{t+s}) - \upsilon(y_{t+s})\right]. \tag{4}$$

This reduced participation constraint shows that higher future output in $[0, y^*]$ relaxes the participation constraint allowing for larger current output. In the following we use $y^{\max}$ to denote the highest, stationary level of output consistent with (4). It is the unique positive solution to $\lambda r \upsilon(y^{\max}) = \alpha \left[u(y^{\max}) - \upsilon(y^{\max})\right]$.

Proposition 1 ***(cea under pairwise meetings)***

1. *If $y^* \leq y^{\max}$, then any cea is such that $y_t = y^*$ and $\ell_t \in [\upsilon(y^*), \overline{\ell}]$ for all $t \in \mathbb{N}_0$, where $\overline{\ell} = \alpha[u(y^*) - \upsilon(y^*)]/\lambda r$.*

2. *If $y^* > y^{\max}$, then the cea is such that $y_t = y^{\max}$ and $\ell_t = \upsilon(y_t)$ for all $t \in \mathbb{N}_0$.*

If agents are sufficiently patient (r low) and if the temptation to renege is not too large (λ low), then first-best allocations are implementable. The equilibrium size of ℓ is indeterminate when $y^* < y^{\max}$, and we give an upper bound, $\overline{\ell}$, for it (which may not be achievable every period). In contrast, if $\lambda r > \alpha \left[u(y^*)/\upsilon(y^*) - 1\right]$, then the cea is $y_t = y^{\max} < y^*$, which corresponds to the highest steady state.

We now turn to the implementation of the cea with an explicit bargaining protocol and solvency constraints. Suppose buyers set the terms of the loan contract unilaterally and cannot promise to repay more

than some endogenous debt limit, d_t. The value function of a buyer solves

$$V_t^b = \alpha \max_{y_t, \ell_t} \left\{ u(y_t) - \ell_t + \beta V_{t+1}^b \right\} + (1-\alpha)\beta V_{t+1}^b \quad \text{s.t. } \ell_t = \upsilon(y_t) \leq d_t. \tag{5}$$

With probability α, a buyer is matched in stage 1 in which case he extends an offer, (y_t, ℓ_t), that makes the seller indifferent between accepting and rejecting, $\ell_t - \upsilon(y_t) = 0$. The offer must also satisfy a solvency constraint according to which the buyer cannot promise to repay more than d_t. The solution is $y_t = \min\left\{y^*, \upsilon^{-1}(d_t)\right\}$. The highest debt limit that is consistent with repayment, the "not-too-tight" solvency constraint of AJ, solves $\lambda d_t = \beta V_{t+1}^b$. In a stationary equilibrium it solves $r\lambda d = \alpha\left[u(y) - v(y)\right]$ where $y = \min\left\{y^*, \upsilon^{-1}(d)\right\}$. If $r\lambda v(y^*) \leq \alpha\left[u(y^*) - v(y^*)\right]$ then $d \geq v(y^*)$ and $y = y^* \leq y^{\max}$. If $r\lambda v(y^*) > \alpha\left[u(y^*) - v(y^*)\right]$ then $d = v(y^{\max}) < v(y^*)$ and $y = y^{\max} < y^*$, which corresponds to cea's in Proposition 1. Hence, we have the following implementation result.

Proposition 2 ***(Second Welfare Theorem for economies with pairwise meetings)*** *The cea is implemented with take-it-or-leave-it offers by buyers under "not-too-tight" solvency constraints.*

Notice that the cea is not uniquely implemented by the optimal mechanism. Indeed, under buyer take-it-or-leave-offer, any sequence $\{d_t\}$ that satisfies

$$d_t = \beta\{\alpha[u(y_{t+1}) - v(y_{t+1})] + d_{t+1}\} \tag{6}$$

with $y_{t+1} = \min\{\upsilon^{-1}(d_{t+1}), y^*\}$ corresponds to an equilibrium under "not-too-tight" solvency constraints. It is straightforward to check that there are a continuum of equilibria satisfying (6), and any such equilibrium with $d_0 < v(y^{\max})$ converges to the autarky steady state. The cea is the only bounded sequence that does not converge to the autarky equilibrium.[6]

3.2 Optimal mechanism with large-group meetings

Suppose next that a fraction α of buyers and sellers meet together in a centralized location. If we only impose individual rationality, then the planner's problem is subject to the same incentive constraints as before, (2) and (3), and Proposition 1 holds. However, the restriction according to which no coalition of agents within a meeting can defect from the proposed allocation is binding when $\upsilon'' > 0$. Indeed, suppose the allocation is as in Proposition 1 where each seller receives $v(y)$ for producing y. Now a buyer and two sellers can form a deviating coalition in which each seller produces $y_t/2$ at a total cost of $2\upsilon(y_t/2) < \upsilon(y_t)$ and the buyer compensates the sellers by offering them a positive surplus. In order to prevent such defections we impose the core requirement or, equivalently, the competitive equilibrium outcome as in Kehoe and Levine (1993)

[6] For related results in the context of the AJ model see Bloise et al. (2013).

and AJ.[7] Matched agents take the price of stage-1 goods in terms of stage-2 goods, p, as given, and by seller's optimization, $p = v'(y)$. Hence, in exchange for consuming y of the stage-1 good the buyer must deliver $\eta(y) = v'(y)y$ of the stage-2 good. This core requirement also implies that $y \leq y^*$.

We are now ready to formulate the planner's problem analogous to (1)-(3). The objective function is still (1). The reduced participation constraint, (4), now becomes

$$\lambda\eta(y_t) \leq \sum_{s=1}^{+\infty} \beta^s \alpha \left[u(y_{t+s}) - \eta(y_{t+s})\right], \tag{7}$$

and we have the requirement that $y_t \leq y^*$ for all t. In supplemental appendix $S2$ we show that we can reformulate this problem in a recursive manner, by introducing the buyer's "guaranteed minimum utility," ω_t, as a new state variable.[8] It is shown there that society's welfare, denoted $V(\omega)$, solves the following Bellman equation,

$$\begin{aligned} V(\omega) &= \max_{y,\omega'} \left\{\alpha\left[u(y) - v(y)\right] + \beta V(\omega')\right\} && (8)\\ \text{s.t.} \quad & -\eta(y) + \beta\frac{\omega'}{\lambda} \geq 0 && (9)\\ & \omega' \geq (1+r)\left\{\omega - \alpha\left[u(y) - \eta(y)\right]\right\} && (10)\\ & y \in [0, y^*],\ \omega' \in [0, \bar{\omega}], && (11) \end{aligned}$$

where $\bar{\omega} = \alpha \max_{y\in[0,y^*]} \left[u(y) - \eta(y)\right]/(1-\beta)$ is an upper bound for the lifetime expected utility of a buyer across all incentive feasible allocations. This upper bound is computed by assigning the maximum surplus to buyers in every period. This allocation, however, might not satisfy the stage-2 participation constraint, (9). This incentive constraint is derived from the participation constraint, (7), and it says a buyer is better off delivering $\eta(y)$ units of stage-2 good so that he can enjoy the future utility from the proposed allocation, $\beta\omega'$, rather than consuming his own output, which was intended for sellers' consumption, and enjoy utility $\lambda\eta(y)$. The novelty is the promise-keeping constraint, (10), according to which the guaranteed minimum lifetime expected utility promised to a buyer along the equilibrium path, ω, is implemented by generating an expected surplus in the current period equal to $\alpha\left[u(y) - \eta(y)\right]$ and by promising $\beta\omega'$ for the future.

We define two critical values for stage-1 output:

$$\begin{aligned} \hat{y} &= \arg\max_{y\in[0,y^*]} [u(y) - \eta(y)] && (12)\\ y^{\max} &= \max\{y > 0 : \alpha[u(y) - \eta(y)] \geq r\lambda\eta(y)\}. && (13) \end{aligned}$$

[7] See Wallace (2013) for a related assumption in the context of monetary economies. The equivalence result between the core and competitive equilibrium allocations for economies with a continuum of agents was first shown by Aumann (1964). See supplementary appendix $S1$ for a proof of this equivalence result in the context of our model.

[8] Our recursive formulation is similar to the self-generation technique in Abreu et al. (1990), which characterizes the set of payoffs generated by Perfect Public Equilibria.

The quantity $\hat{y}$ is the output level that maximizes the buyer's surplus in stage 1. The quantity $y^{\max}$ is the highest, stationary level of output that is consistent with the buyer's participation constraint. We assume that both $\hat{y}$ and $y^{\max}$ are well-defined and, for all $0 \leq y \leq y^{\max}$, $\alpha[u(y) - \eta(y)] \geq r\lambda\eta(y)$.

Proposition 3 *Suppose that $y^* > y^{\max} > \hat{y}$. There is a unique solution to (8)-(11) in the space of continuous and bounded functions. It is weakly decreasing in ω and concave provided that η is convex.*

Using that V is weakly decreasing in ω and the fact that ω_0 is a choice variable of the planner, the maximum value for society's welfare is $V(0) = \max_{\omega \in [0,\bar{\omega}]} V(\omega)$. This gives us the following Proposition.

Proposition 4 ***(cea under centralized meetings)*** *Assume η is a convex function.*

1. *If $y^* \leq y^{\max}$, then the cea is such that $y_t = y^*$ for all $t \in \mathbb{N}_0$.*
2. *If $y^{\max} \leq \hat{y} \leq y^*$, then the cea is such that $y_t = y^{\max}$ for all $t \in \mathbb{N}_0$.*
3. *If $\hat{y} < y^{\max} < y^*$ then there are two cases:*
   (a) *If $\lambda \geq \alpha\left[1 - u'(y^{\max})/\eta'(y^{\max})\right]$, then the cea is such that $y_t = y^{\max}$ for all $t \in \mathbb{N}_0$.*
   (b) *If $\lambda < \alpha\left[1 - u'(y^{\max})/\eta'(y^{\max})\right]$, then the cea is such that $y_0 \in (y^{\max}, y^*)$ and $y_t = y_1 \in (\hat{y}, y^{\max})$ for all $t \geq 1$, where (y_0, y_1) is the unique solution to*

$$\max_{y_0, y_1} \left\{ u(y_0) - v(y_0) + \frac{u(y_1) - v(y_1)}{r} \right\} \tag{14}$$

$$s.t. \quad \eta(y_0) = \frac{\alpha[u(y_1) - \eta(y_1)]}{\lambda r}. \tag{15}$$

Provided that agents are sufficiently patient, $r \leq \alpha[u(y^*) - \eta(y^*)]/\lambda\eta(y^*)$, the cea coincides with a first-best allocation. Note that the condition to implement the first best is more stringent than the one under pairwise meetings since $\eta(y) \geq v(y)$, with a strict inequality when $v'' > 0$. If $y^{\max} < y^*$ then the first best violates the buyers' participation constraint, in which case the characterization of the cea depends on the ordering of $y^{\max}$ and $\hat{y}$. As shown in the left panel of Figure 1, if $y^{\max} \leq \hat{y}$ then a buyer's intra-period surplus and society's welfare are both increasing with y over $(0, y^{\max})$. Hence, by raising buyers' consumption in the future the planner relaxes the current participation constraint allowing for more consumption today. So the highest steady state, $y = y^{\max}$, is constrained efficient.

We now turn to the case where $\hat{y} < y^{\max} < y^*$. For all $y \in (\hat{y}, y^*)$ a buyer's surplus, $u(y) - \eta(y)$, decreases with y whereas society's surplus, $u(y) - v(y)$, increases with y, as illustrated in the right panel of Figure 1. A buyer's surplus decreases for $y > \hat{y}$ because of a pecuniary externality according to which an

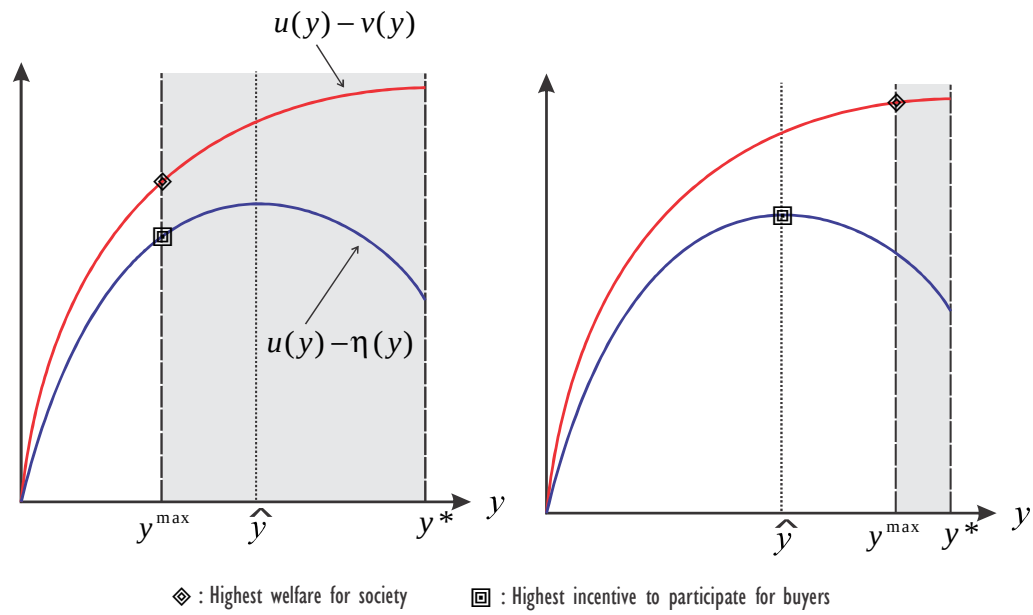

Figure 1: Let panel: No trade-off between efficiency and incentives over $[0, y^{\max}]$; Right panel: A trade-off between efficiency and incentives over $[\hat{y}, y^*]$.

increase in aggregate consumption raises the price of stage-1 goods, $p = v'(y)$, which lowers $u(y) - py$ when y is sufficiently close to y^*.[9] The negative relationship between buyers' surplus and aggregate consumption gives rise to an intertemporal trade-off between current and future output. Indeed, in order to raise current buyers' consumption the planner must relax the repayment constraint, which requires lower future prices and hence lower aggregate consumption in the future. As a result of this trade-off, the highest steady state, $y^{\max}$, might no longer be the solution to the planner's problem.

A key result allowing us to characterize the cea in closed form is that it is always socially optimal to keep future output constant, $y_t = y_1$ for all $t \geq 1$. Indeed, we prove that whenever the utility promised to buyers is $\omega > \omega^{\max} = \alpha[u(y^{\max}) - \eta(y^{\max})]/(1-\beta)$, the repayment constraint is slack and it is optimal to set $\omega' = \omega$, which is achieved by keeping consumption constant. In order to establish that $\omega_1 \geq \omega^{\max}$ we use a guess-and-verify method allowing us to compute $V(\omega)$ in closed-form. If $\omega \in (\omega^{\max}, \bar{\omega}]$, $\omega' = \omega$ and from (8)

$$V(\omega) = \frac{\alpha\left[u(y) - v(y)\right]}{1-\beta} \quad \text{with } \alpha\left[u(y) - \eta(y)\right] = (1-\beta)\omega.$$

Moreover, we show that $V(\omega) = V(\omega^{\max})$ for all $\omega < \omega^{\max}$ when $\lambda \geq \alpha\left[1 - u'(y^{\max})/\eta'(y^{\max})\right]$. Figure 2 plots $V(\omega)$ for three cases in Proposition 4. When $y^{\max} \leq \hat{y} \leq y^*$ (case 2), $V(\omega)$ is constant and the cea is stationary at $y_t = y^{\max}$ and $\omega_t = \omega^{\max}$ for $t \geq 1$. As λ decreases (cases 3a and 3b), the temptation to renege

[9] The existence of such pecuniary externality in monetary economies with competitive trades and their welfare implications are discussed in Rocheteau and Wright (2005) and Berentsen, Huber, and Marchesiani (2014, 2016).

is lower and $V(\omega)$ shifts upward. Moreover, it is downward sloping for $\omega > \omega^{\max}$.

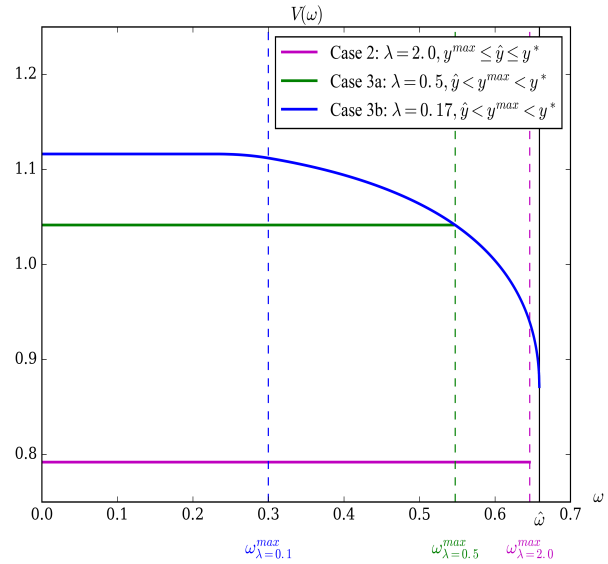

Figure 2: Value functions for $\lambda \in \{2.0, 0.5, 0.17\}$, $\alpha = 1$, $\beta = 0.4$, $u(y) = y$, $v(y) = 0.32 * y^{3.1}$.

If the first best cannot be achieved, a buyer's participation constraint at $t = 0$ must be binding since otherwise y_0 could be raised without affecting any future incentive constraints. As a result the buyer's participation constraint at $t = 0$ is given by (15) and the magnitude of the trade-off between current and future output in the neighborhood of the highest steady state is:

$$\left.\frac{dy_1}{dy_0}\right|_{y^{\max}} = \frac{\lambda r \eta'(y^{\max})}{\alpha[u'(y^{\max}) - \eta'(y^{\max})]} < 0.$$

When $\lambda \geq \alpha\left[1 - u'(y^{\max})/\eta'(y^{\max})\right]$ exploiting this trade-off is harmful since one would have to implement a large drop in future output in order to raise current output by a small amount while maintaining a buyer's incentive to deliver the stage-2 good.

In contrast, when λ is small, it is optimal to exploit the trade-off between current and future output arising from (15). The optimal allocation is such that $y_0 > y^{\max}$ while $y_1 < y^{\max}$. The initial period is interpreted as a credit boom and future periods correspond to a stagnation phase. Even though, in future periods, society would be better-off at the highest steady state, $u(y_1) - v(y_1) < u(y^{\max}) - v(y^{\max})$, buyers enjoy a higher surplus, $u(y_1) - \eta(y_1) > u(y^{\max}) - \eta(y^{\max})$, which relaxes their incentive constraint for repayment at $t = 0$. As a result, output and society's welfare in the initial period are higher than the highest steady-state levels, $u(y_0) - v(y_0) > u(y^{\max}) - v(y^{\max})$.[10]

[10] Kehoe and Levine (1993) provide an example where partial exclusion leads to a welfare-improving outcome. See their Example 2 on p. 875.

In Figure 3 we illustrate the determination of (y_0, y_1). The red curve labeled IR corresponds to (15). It slops downward because of the trade-off between current and future output described above. By definition the IR curve intersects the 45°-line at $y^{\max}$. The blue curve labeled FOC corresponds to the first-order condition of the problem (14)-(15). Given the strict concavity of the surplus function it is optimal to smooth consumption by increasing y_0 when y_1 increases. When λ is low the FOC curve is located above the IR curve at $y_1 = y^{\max}$. Hence, the optimal solution, denoted (y_0^{**}, y_1^{**}), is such that $y_0^{**} > y^{\max}$ and $y_1^{**} < y^{\max}$.

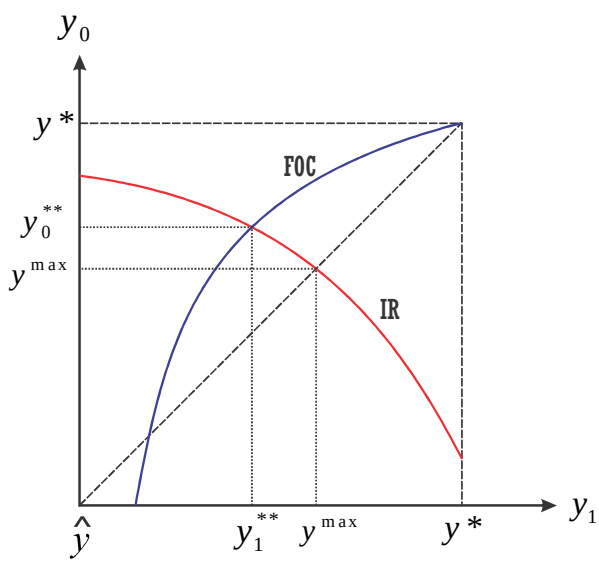

Figure 3: Determination of the constrained-efficient allocation, (y_0, y_1)

We now turn to the implementation of the cea with competitive trades and solvency constraints. The value function of a buyer solves

$$V_t^b = \alpha \max_{p_t y_t \leq d_t} \left\{ u(y_t) - p_t y_t + \beta V_{t+1}^b \right\} + (1-\alpha)\beta V_{t+1}^b, \tag{16a}$$

where $p_t = v'(y_t)$ by market clearing. The solution is such that $y_t = y^*$ if $v'(y^*)y^* \leq d_t$ and $v'(y_t)y_t = d_t$ otherwise. The sequence of debt limits, $\{d_t\}$, must satisfy $\lambda d_t \leq \beta V_{t+1}^b$. Bethune et al. (2017) specifies strategies that sustain any such sequence $\{d_t\}$ as an equilibrium outcome. Solvency constraints are said to be "not-too-tight" when the previous inequality holds at equality. In order to implement $y = y^{\max}$ in all periods the debt limit must coincide with the "not-too-tight" solvency constraint, i.e., it solves $\lambda r d = \alpha \{[u(y) - v'(y)y]\}$. If the cea is non-stationary then (y_0, y_1) is implemented with debt limits $d_0 = v'(y_0)y_0$ and $d_1 = v'(y_1)y_1$ where $d_1 < d_0 = \alpha \{[u(y_1) - v'(y_1)y_1]\} / \lambda r$. The following implementation result then follows from Proposition 4.

Proposition 5 ***(Second Welfare Theorem under large-group meetings)*** *Assume that η is a convex*

function.

1. *If either* $y^{\max} \leq \hat{y} \leq y^*$ *or* $\hat{y} < y^{\max} < y^*$ *and* $\lambda \geq \alpha\left[1 - u'(y^{\max})/\eta'(y^{\max})\right]$*, then the cea is implemented with "not-too-tight" solvency constraints.*

2. *If* $\hat{y} < y^{\max} < y^*$ *and* $\lambda < \alpha\left[1 - u'(y^{\max})/\eta'(y^{\max})\right]$*, then the cea is implemented with slack repayment constraints (i.e., "too-tight" solvency constraints) in all future periods,* $t \geq 1$*.*

The failure of the AJ Welfare Theorem in the second part of the Corollary is surprising as one would conjecture that higher debt limits allow society to generate larger gains from trade. This reasoning is valid in a static sense. If d_t increases, the sum of all surpluses in period t, $\alpha\left[u(y_t) - v(y_t)\right]$, increases. However, there is a general equilibrium effect according to which an increase in d_t raises the price of stage-1 goods. If the economy is close enough to the first best, this pecuniary externality lowers buyers' welfare (even though society as a whole is better off) and worsens their incentive to repay their debt in earlier periods.

The conditions in the second part of the Corollary are satisfied for all the numerical examples in GMMW. However, GMMW impose "not-too-tight" solvency constraints as part of their equilibrium solution. Our results show that these constraints fail to implement cea's and hence the equilibrium cycles in GMMW are dominated in terms of ex-ante welfare by equilibria with "too-tight" solvency constraints.

3.3 Optimal stochastic credit cycles

If the cea characterized in Proposition 4 is non-stationary, then it is also time-inconsistent in the following sense: if the planner were to re-optimize at a later date, when the economy is in permanent stagnation, it would want to deviate and generate a new credit boom. In order to address this time inconsistency we now study equilibria where the realization of a sunspot state allows the planner to re-optimize. We think of the planner as having partial (or loose) commitment.[11] This sunspot occurs at the beginning of a period with probability γ, where γ parameterizes the strength of the planner's commitment. The planner's problem becomes:

$$V(\omega) = \max_{y,\omega'} \left\{\alpha\left[u(y) - v(y)\right] + \beta\left[(1-\gamma)V(\omega') + \gamma V(0)\right]\right\} \tag{17}$$

$$\text{s.t.} \quad -\eta(y) + \beta\frac{\left[(1-\gamma)\omega' + \gamma\omega_0\right]}{\lambda} \geq 0 \tag{18}$$

$$(1-\gamma)\omega' + \gamma\omega_0 \geq (1+r)\left\{\omega - \alpha\left[u(y) - \eta(y)\right]\right\} \tag{19}$$

$$y \in [0, y^*],\ \omega' \in [0, \bar{\omega}], \tag{20}$$

[11] This approach is analogous to the formalization of partial or loose commitment by Debortolia and Nunes (2010) and Kovrijnykh (2013).

where ω_0 is the lifetime expected utility of a buyer at the beginning of a cycle when the sunspot state is realized, i.e.,

$$\omega_0 = \frac{[1-\beta(1-\gamma)]\sum_{\tau=0}^{+\infty}[\beta(1-\gamma)]^{\tau}[u(y_\tau)-\eta(y_\tau)]}{1-\beta}, \tag{21}$$

where $\{y_\tau\}$ is stage-1 consumption chosen by the planner. Notice that when choosing $\{y_t, \omega_{t+1}\}$ the planner takes ω_0 as given, i.e., he takes the choice of his future-self when he has the opportunity to reoptimize as given. Hence, the solution to the planner's problem under partial commitment involves solving a fixed point problem. According to (17), the planner is committed to his promise of future utility, ω', with probability $1-\gamma$. With probability γ the planner reoptimizes, in which case its continuation value is $V(0) = V(\omega_0)$ and the buyer's expected utility is ω_0. According to (18), a buyer who reneges on his obligation to repay $\eta(y)$ gives up the expected continuation value $(1-\gamma)\omega' + \gamma\omega_0$. The promise-keeping constraint, (19), states that a buyer's discounted expected utility, $\beta[(1-\gamma)\omega' + \gamma\omega_0]$, must be equal to the initial promise net of the expected surplus in the current period. Finally, taking ω_0 as given, the highest expected utility a buyer can be promised is

$$\bar{\omega} = \frac{\max_{y\in[0,y^*]}[u(y)-\eta(y)] + \beta\gamma\omega_0}{1-\beta(1-\gamma)}. \tag{22}$$

Given ω_0, $V(\omega)$ can be computed by iterations of value functions. The policy function gives $\{y_\tau\}$ which allows us to update ω_0 and re-compute $V(\omega)$. We iterate this procedure until convergence, i.e., until we find a fixed point.

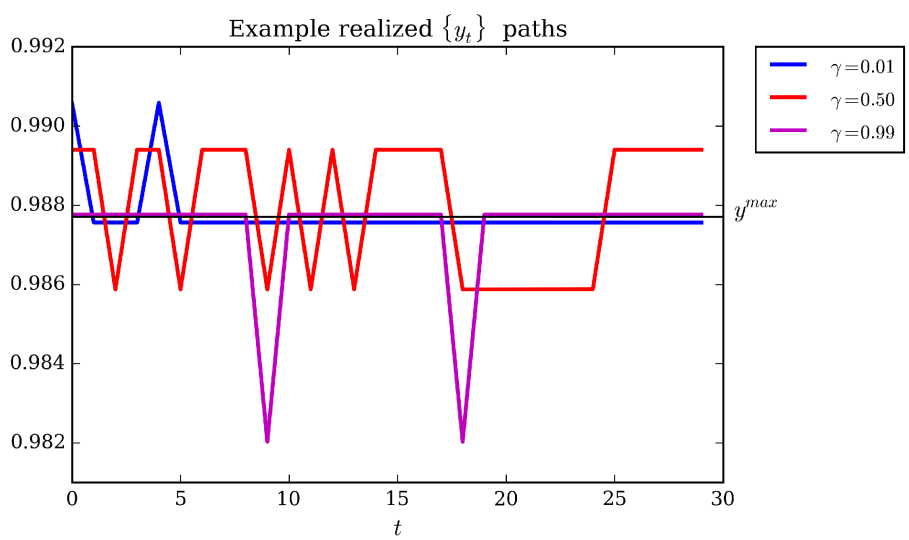

Figure 4: Optimal sunspot equilibria

In Figure 4 we illustrate the solution to the planner's problem for the parametrization $\lambda = 0.1$, $\alpha = 0.1$, $u(y) = y$, $v(y) = y^{1.5}/1.5$, and for $\gamma \in \{0.01.0.5, 0.99\}$. For each γ we generate a sequence of realizations for

the sunspot state and we plot the associated path for stage-1 output. The solution to the planner's problem is qualitatively similar to the one described in Proposition 4. At the start of the cycle, when the sunspot state is realized, output increases to some level $y_0 > y^{\max}$. For all subsequent periods until the next sunspot realization, output is low at $y_1 < y^{\max}$. As the stochastic cycle repeats itself, the economy alternates through credit booms and busts of random lengths.

As γ approaches 1, e.g., $\gamma = 0.99$ in Figure 4, y_0 converges to $y^{\max}$. If the commitment power is very low, credit booms are small but they happen in almost all periods. From an outside observer, these booms are normal times. When the sunspot is not realized, which happens 1% of the time, output falls sharply and the economy goes through a credit crunch. With low commitment, the economy appears to experience rare but pronounced credit crunches.

As γ approaches 0, e.g., $\gamma = 0.01$ in Figure 4, (y_0, y_1) converges to the "full commitment" cea. In the infrequent event when the sunspot occurs, the economy experiences a credit boom and output increases above $y^{\max}$. Following the boom, output falls slightly below $y_{\max}$. So with high commitment, the economy experiences rare but large credit booms.

For intermediate values for γ, e.g., $\gamma = 0.5$ in Figure 4, y_0 is significantly larger than $y^{\max}$ while y_1 is significantly lower than $y^{\max}$ and both output levels occur frequently. The equilibrium features optimal (given γ) business cycle fluctuations driven by endogenous changes in the availability of credit.

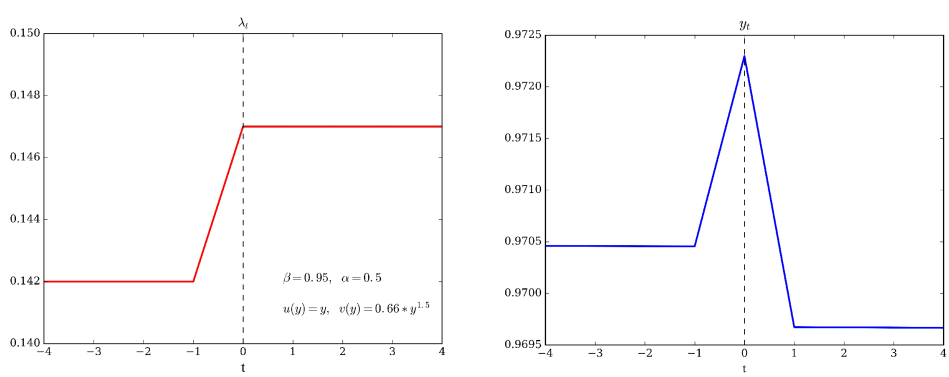

Figure 5: Unanticipated increase in λ

So far the state allowing agents to coordinate on an equilibrium does not affect fundamentals – it is pure extrinsic uncertainty. Alternatively, the planner could reoptimize – i.e., agents could coordinate on a new equilibrium – following the realization of real shocks. As an example, consider an unexpected increase in the preference parameter λ at $t = 0$ from $\lambda_{0^-} = 0.142$ to $\lambda_{0^+} = 0.147$ as illustrated in Figure 5 for $\beta = 0.95$, $u(y) = y$, $v(y) = \sqrt{y}$, and $\alpha = 0.5$. The change in fundamentals makes the incentive problem more severe.

For all $t < 0$, $y_t = y_1$ corresponding to the initial cea. At $t = 0$ agents coordinate on the constrained-efficient equilibrium corresponding to the new value of λ. Output and debt levels increase initially and then they fall to a permanently lower level. So there is a temporary credit boom followed by a permanent credit bust and a permanent reduction in output below the initial steady state. These allocations are optimal at time $t = 0$.

4 Conclusion

We asked whether extrinsic credit fluctuations could be socially optimal. To answer this question we characterized constrained-efficient allocations of a two-good, pure credit economy under limited commitment. We showed that if agents interact through random, pairwise meetings, then constrained-efficient allocations are constant through time and exhibit the highest output level that is incentive feasible. Such allocations can be implemented with take-it-or-leave-it offers by buyers/borrowers and "not-too-tight" solvency constraints. If agents meet in a centralized location, thereby allowing for a larger set of feasible deviations, then the constrained-efficient allocation can be non-stationary. In such cases optimal allocations are characterized by an initial credit boom with high output followed by a permanent stagnation with low output. Moreover, constrained-efficient allocations are not implemented with "not-too-tight" solvency constraints, in contrast to AJ and GMMW. Indeed, it is optimal to have slack participation constraints in all future periods by restricting borrowing in order to lower future prices and relax the repayment constraint in the initial period. Because the planner's solution is time-inconsistent — the planner would like to renege on the stagnation and create a new credit boom — we also solved the planner's problem under various degrees of commitment. If commitment is weak, the economy experiences rare but pronounced credit crunches. If commitment is strong, then the economy experiences infrequent but large credit booms.

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Appendix: Proofs of propositions

Proof of Proposition 1. (1) Suppose that $y^* \leq y^{\max}$. Then, the outcome $\{(y_t, \ell_t)\}_{t=0}^{\infty}$ with $y_t = y^*$ and $\ell_t = v(y_t)$ for all t is implementable, which is the first-best and hence is cea. Now, suppose that $\{(y_t, \ell_t)\}_{t=0}^{\infty}$ is a cea. Then, $y_t = y^*$ for all t, and since equilibrium requires $\ell_t \geq v(y_t)$ for all t, (2) implies that

$$\lambda \ell_t \leq \sum_{s=1}^{+\infty} \beta^s \alpha[u(y_{t+s}) - v(y_{t+s})] = \frac{1}{r}\alpha[u(y^*) - v(y^*)],$$

and hence $\ell_t \leq \overline{\ell}$.

(2) Suppose that $y^* > y^{\max}$. We show that the optimal sequence that has $y_t = y^{\max}$ and $\ell_t = v(y_t)$ for all t. Suppose, by contradiction, that there is another sequence $\{y'_t, \ell'_t\}_{t=0}^{\infty}$ satisfying (2) and (3) with a strictly higher welfare. It then follows that $y^* \geq y'_t > y^{\max}$ for some t. Let t_0 be the first t such that $u(y'_t) - v(y'_t) > u(y^{\max}) - v(y^{\max})$. Now we show that for some $t_1 > t_0$, $y'_{t_1} > y'_{t_0}$. Suppose, by contradiction, that $y'_t \leq y'_{t_0}$ for all $t > t_0$. We have the following inequality,

$$v(y'_{t_0}) \leq \ell'_{t_0} \leq \lambda^{-1} \sum_{s=1}^{+\infty} \beta^s \alpha \left[u(y'_{t_0+s}) - \ell'_{t_0+s}\right] \leq \lambda^{-1} \sum_{s=1}^{+\infty} \beta^s \alpha \left[u(y'_{t_0}) - v(y'_{t_0})\right],$$

where the first inequality follows from the seller's participation constraint, (3), at $t = t_0$, the second follows from the buyer's participation constraint, (2), and the third follows from $u(y'_{t_0+s}) - \ell'_{t_0+s} \leq u(y'_{t_0+s}) - v(y'_{t_0+s}) \leq u(y'_{t_0}) - v(y'_{t_0})$ since $u - v$ is increasing for $y < y^*$ and $\ell'_{t_0+s} \leq v(y'_{t_0+s})$ for all s. Because $y^{\max}$ is the maximal value of y'_{t_0} that equalizes the left side and the right side of this series of inequalities, it follows that $y'_{t_0} \leq y^{\max}$, a contradiction. So $y^* \geq y'_{t_1} > y'_{t_0}$ for some t_1 (and we choose $t_1 > t_0$ to be the first index for this to happen). By induction, we can then find a subsequence $\{y'_{t_i}\}$ that is strictly increasing and is bounded from above. So there exists a limit $\widetilde{y} = \lim_{i\to\infty} y'_{t_i} > y^{\max}$. Hence, by monotonicity, we have for all i,

$$rv(y'_{t_i}) \leq r\ell_{t_i} \leq \frac{\alpha[u(\widetilde{y}) - v(\widetilde{y})]}{\lambda},$$

and, by taking i to infinity, we have

$$rv(\widetilde{y}) \leq \frac{\alpha[u(\widetilde{y}) - v(\widetilde{y})]}{\lambda}.$$

However, as explained above, this implies that $\widetilde{y} \leq y^{\max}$, and this leads to a contradiction.

Proof of Proposition 3. First we prove that there is a unique solution to the Bellman equation. We begin by showing that, for any $\omega \in [0, \bar{\omega}]$, the set of elements $(y, \omega') \in [0, y^*] \times [0, \overline{\omega}]$ satisfying (9)-(11) is nonempty and hence the maximization problem is well-defined. For all $\omega \in [0, \bar{\omega}]$, define $y_\omega \leq \hat{y} \leq y^*$ as the

unique solution to

$$\omega = \frac{\alpha}{1-\beta}[u(y_\omega) - \eta(y_\omega)]. \tag{23}$$

As $u(0) - \eta(0) = 0$ and $\frac{\alpha}{1-\beta}[u(\hat{y}) - \eta(\hat{y})] = \overline{\omega}$, such $y_\omega \in [0, \hat{y}]$ exists by the Intermediate Value Theorem. We claim that (y_ω, ω') satisfies (9)-(11) for any $\omega' \in [\omega, \bar{\omega}]$. First (11) holds by construction. Moreover, rearranging (23), we have

$$\beta\omega = \omega - \alpha[u(y_\omega) - \eta(y_\omega)]$$

which implies (10) for any $\omega' \geq \omega$. Finally, by (23) and the fact that $y \leq \hat{y} \leq y^{\max}$,

$$\eta(y_\omega) \leq \beta\frac{\omega}{\lambda} \leq \beta\frac{\omega'}{\lambda}$$

for any $\omega' \geq \omega$.

Proof. We now show that the Bellman equation (8)-(11) has a unique solution. Let $\mathcal{C}[0, \bar{\omega}]$ be the complete metric space of continuous functions over $[0, \bar{\omega}]$ equipped with the sup norm. Define $T : \mathcal{C}[0, \bar{\omega}] \to \mathcal{C}[0, \bar{\omega}]$ by

$$T(W)(\omega) = \max_{y,\omega'} \{\alpha\,[u(y) - v(y)] + \beta W(\omega')\}\,,$$

subject to (9)-(11). Note that $T(W) \in \mathcal{C}[0, \bar{\omega}]$ by the Theorem of Maximum. The mapping T satisfies the Blackwell sufficient condition (Lucas and Stokey, 1989, Theorem 3.3), and hence T is a contraction mapping, which admits a unique fixed point by the Banach Fixed-Point Theorem. Hence, V is the unique solution to the Bellman equation and is continuous.

Notice that by decreasing ω we increase the set of (y, ω') that satisfies (9)-(11), but without affecting the objective function. Hence, V is weakly decreasing.

Now we prove that V is weakly decreasing in ω, assuming that η is convex. To show that V is concave, we show that T preserves concavity. Let $\omega_0, \omega_1 \in [0, \bar{\omega}]$ be given. Let (y_0, ω_0) and (y_1, ω_1) solves (9)-(11) for ω_0 and ω_1, respectively. let $\epsilon \in (0, 1)$ be given. Then,

$$\begin{aligned}
& T(W)(\epsilon\omega_0 + (1-\epsilon)\omega_1) \\
\geq\ & \alpha\,[u(\epsilon y_0 + (1-\epsilon)y_1) - v(\epsilon y_0 + (1-\epsilon)y_1)] + \beta W(\epsilon\omega_0' + (1-\epsilon)\omega_1') \\
\geq\ & \alpha\epsilon[u(y_0) - v(y_0)] + \alpha(1-\epsilon)[u(y_0) - v(y_0)] + \beta[\epsilon W(\omega_0') + (1-\epsilon)W(\omega_0')] \\
=\ & \epsilon T(W)(\omega_0) + (1-\epsilon)T(W)(\omega_1).
\end{aligned}$$

The first inequality follows from the fact that $(\epsilon y_0 + (1-\epsilon)y_1, \epsilon\omega_0' + (1-\epsilon)\omega_1')$ also satisfies (9)-(11) for $\omega = \epsilon\omega_0 + (1-\epsilon)\omega_1$ because η is convex. The second inequality follows from the concavity of $u - v$ and the assumed concavity of W. ■

Proof of Proposition 4. The program that selects the best PBE is

$$\max_{\{y_t\}_{t=0}^{\infty}} \sum_{t=0}^{+\infty} \beta^t \alpha \left[u(y_t) - \upsilon(y_t)\right] \tag{24}$$

$$\text{s.t.} \qquad \lambda\eta(y_t) \leq \alpha \sum_{s=1}^{+\infty} \beta^s \left[u(y_{t+s}) - \eta(y_{t+s})\right] \tag{25}$$

$$y_t \leq y^* \text{ for all } t = 0, 1, 2, ... \tag{26}$$

(1) Suppose that $y^* \leq y^{\max}$. In this case, the outcome $\{y_t\}_{t=0}^{\infty}$ with $y_t = y^*$ for all t is implementable and hence is the cea

(2) Suppose that $y^* > y^{\max}$ but $y^{\max} \leq \hat{y}$. We show that the outcome $\{y_t\}_{t=0}^{\infty}$ with $y_t = y^{\max}$ for all t is the optimum. Suppose, by contradiction, that there is another outcome $\{y_t'\}_{t=0}^{\infty}$ satisfying (25) and (26) with a strictly higher welfare. First we show that $y_t' \leq \hat{y}$ for all t. Suppose, by contradiction, that there is a t such that $y_t' > \hat{y}$. Then, because $\hat{y} \geq y^{\max}$,

$$\lambda\eta(y_t') > \lambda\eta(\hat{y}) \geq \sum_{s=1}^{\infty} \beta^s \alpha[u(\hat{y}) - \eta(\hat{y})] \geq \sum_{s=1}^{\infty} \beta^s \alpha[u(y_{t+s}') - \eta(y_{t+s}')],$$

a contradiction to (25). Given that this alternative outcome can only lie in the range $[0, \hat{y}]$ and hence the trade surplus is increasing in the output, the rest of the arguments are exactly the same as those in the proof of Proposition 1.

(3) Suppose that $\hat{y} < y^{\max} < y^*$. As mentioned, we can solve the planner's problem by solving the Bellman equation (8)-(11) and by choosing ω_0. Moreover, by Proposition 3, the value function V is unique, and it is nonincreasing and concave. Hence, we may choose $\omega_0 = 0$. After solving the Bellman equation and the policy functions, denoted by $\omega'(\omega_t)$ and $y(\omega_t)$, the cea can then by computed by $\omega_{t+1} = \omega'(\omega_t)$, $y_t = y(\omega_t)$, and $\ell_t = \eta(y_t)$.

The Lagrangian associated with the Bellman equation (8)-(11) is

$$\begin{aligned} \mathcal{L} \quad = \quad & \alpha \left[u(y) - v(y)\right] + \beta V(\omega') + \xi \left(\beta \frac{\omega'}{\lambda} - \eta(y)\right) \\ & + \nu \left\{\alpha \left[u(y) - \eta(y)\right] + \beta\omega' - \omega\right\}, \end{aligned} \tag{27}$$

where the Lagrange multipliers, ξ and ν, are non-negative. In general V may not be differentiable everywhere. However, because V is concave, the following first-order conditions are still necessary and sufficient for (y, ω') to be optimal (Clarke (1976), Theorems 1 and 2):

$$\alpha \left[u'(y) - v'(y)\right] - \xi\eta'(y) + \nu\alpha \left[u'(y) - \eta'(y)\right] = 0 \tag{28}$$

$$\beta V_+'(\omega') + \beta \frac{\xi}{\lambda} + \beta\nu \leq 0 \leq \beta V_-'(\omega') + \beta \frac{\xi}{\lambda} + \beta\nu, \tag{29}$$

where $V'_+(\omega') = \lim_{\omega \downarrow \omega'} V'(\omega)$ and $V'_-(\omega') = \lim_{\omega \uparrow \omega'} V'(\omega)$. Both $V'_+(\omega')$ and $V'_-(\omega')$ exist because of concavity. The envelope condition, provided that $V'(\omega)$ exists, is

$$V'(\omega) = -\nu. \tag{30}$$

We define two critical values for the buyer's promised utility:

$$\omega^{\max} = \frac{\alpha\left[u(y^{\max}) - \eta(y^{\max})\right]}{1-\beta} \text{ and } \bar{\omega} = \frac{\alpha[u(\hat{y}) - \eta(\hat{y})]}{1-\beta}.$$

The first threshold is the buyer's life-time expected utility at the highest steady state, while the second is the maximum life-time expected utility achieved by the buyers across all steady states. Note that by the definition of $y^{\max}$, $\eta(y^{\max}) = \beta\omega^{\max}/\lambda$.

(a) $\lambda \geq \alpha\left[1 - u'(y^{\max})/\eta'(y^{\max})\right]$.

The following claim provides conditions under which the constrained-efficient allocation corresponds to the highest steady state. In order to establish this claim, we shows that, for $\omega = 0$ and $\omega = \omega^{\max}$, the optimal solution to the maximization problem in (8)-(11) is $(\omega^{\max}, y^{\max})$.

Claim 1 *If $\hat{y} < y^{\max} < y^*$ and $\lambda \geq \alpha\left[1 - u'(y^{\max})/\eta'(y^{\max})\right]$, then the unique solution to (8)-(11) is*

$$V(\omega) = \frac{\alpha\left[u(y^{\max}) - v(y^{\max})\right]}{1-\beta} \text{ if } \omega \in [0, \omega^{\max}], \tag{31}$$

$$= \frac{\alpha\left\{u[g(\omega)] - v[g(\omega)]\right\}}{1-\beta} \text{ if } \omega \in (\omega^{\max}, \bar{\omega}], \tag{32}$$

where $g(\omega)$ is the unique solution to $\alpha[u(y) - \eta(y)] = (1-\beta)\omega$ for all $\omega \in (\omega^{\max}, \bar{\omega}]$.

The function V given by (31)-(32) is flat in the interval $[0, \omega^{\max}]$ and is strictly concave for all $\omega \in (\omega^{\max}, \bar{\omega})$, and hence is concave overall. To show the strict concavity, we compute $V''(\omega)$ for all $\omega \in (\omega^{\max}, \bar{\omega})$. By the Implicit Function Theorem, we have

$$g'(\omega) = \frac{1-\beta}{\alpha\{u'[g(\omega)] - \eta'[g(\omega)]\}} < 0,$$

and hence

$$V'(\omega) = \frac{u'[g(\omega)] - v'[g(\omega)]}{u'[g(\omega)] - \eta'[g(\omega)]} \tag{33}$$

for all $\omega \in (\omega^{\max}, \bar{\omega})$. Thus,

$$\begin{aligned} V''(\omega) &= \frac{\{u''[g(\omega)] - v''[g(\omega)]\}\{u'[g(\omega)] - \eta'[g(\omega)]\}}{\{u'[g(\omega)] - \eta'[g(\omega)]\}^2} g'(\omega) \\ &+ \frac{-\{u'[g(\omega)] - v'[g(\omega)]\}\{u''[g(\omega)] - \eta''[g(\omega)]\}}{\{u'[g(\omega)] - \eta'[g(\omega)]\}^2} g'(\omega) < 0. \end{aligned}$$

Note that, for all $\omega \in (\omega^{\max}, \bar{\omega})$, $u'[g(\omega)] - \eta'[g(\omega)] < 0$ as $g(\omega) > \hat{y}$ and that $u'[g(\omega)] - v'[g(\omega)] > 0$ as $g(\omega) \leq y^{\max} < y^*$.

To prove that V satisfies (31) and (32), we consider two cases.

(i) Suppose that $\omega \in [0, \omega^{\max}]$. The solution to the maximization problem in (8)-(11) is given by $(\omega', y) = (\omega^{\max}, y^{\max})$. This solution is feasible because $(\omega^{\max}, y^{\max})$ satisfies (10) for all $\omega \leq \omega^{\max}$ and it satisfies (9) at equality. Next, we show that it satisfies (28)-(29) with $\nu = 0$ and

$$\xi = \frac{\alpha[u'(y^{\max}) - v'(y^{\max})]}{\eta'(y^{\max})} > 0.$$

The condition (28) holds by the definition of ξ. To establish (29), first note that $V'_-(\omega^{\max}) = 0$ and

$$V'_+(\omega^{\max}) \equiv \lim_{\omega \downarrow \omega^{\max}} V'(\omega) = \frac{u'(y^{\max}) - v'(y^{\max})}{u'(y^{\max}) - \eta'(y^{\max})}.$$

Thus, $V'_-(\omega^{\max}) + \xi/\lambda > 0$ and the first inequality in (29) holds if and only if

$$\begin{aligned} & V'_+(\omega^{\max}) + \frac{\xi}{\lambda} \\ = \quad & \frac{1}{\lambda} \frac{[u'(y^{\max}) - v'(y^{\max})]}{\eta'(y^{\max})} \left\{ \alpha + \lambda \frac{\eta'(y^{\max})}{u'(y^{\max}) - \eta'(y^{\max})} \right\} \leq 0, \end{aligned}$$

and, because $\hat{y} < y^{\max} < y^*$ and hence $u'(y^{\max}) - v'(y^{\max}) > 0$ and $u'(y^{\max}) - \eta'(y^{\max}) < 0$, the last inequality holds if and only if

$$\alpha \left[\frac{u'(y^{\max})}{\eta'(y^{\max})} - 1 \right] \geq -\lambda,$$

that is, $\lambda \geq \alpha \left[1 - u'(y^{\max})/\eta'(y^{\max})\right]$. This implies V satisfies (31).

(ii) Suppose that $\omega \in (\omega^{\max}, \bar{\omega})$. Here we show that $(\omega', y) = (\omega, g(\omega))$ is the solution to the maximization problem in (8)-(11). This solution is feasible: (10) holds by construction; because $\omega' = \omega = \alpha[u(y) - \eta(y)]/(1-\beta)$ and because $y = g(\omega) \leq y^{\max}$,

$$\lambda \eta(y) \leq \beta \alpha [u(y) - \eta(y)]/(1-\beta),$$

(9) holds. Next, we show that the FOC's (28) and (29) are satisfied by $(\omega', y) = (\omega, g(\omega))$ with $\xi = 0$ and

$$\nu = -\frac{u'[g(\omega)] - v'[g(\omega)]}{u'[g(\omega)] - \eta'[g(\omega)]} > 0.$$

The FOC for y, (28), holds by the definition of ν. The FOC for ω', (29), holds if and only if

$$\nu + V'(\omega) = 0,$$

which holds by (33). Thus, if $\omega_0 = \omega \in (\omega^{\max}, \bar{\omega})$, then the optimal sequence is $(\omega_t, y_t) = (\omega, g(\omega))$ for all t. Hence, $V(\omega)$ is satisfies (32) for all $\omega \in (\omega^{\max}, \bar{\omega})$. Finally, V satisfies (32) at $\omega = \bar{\omega}$ by continuity.

(b) $\lambda < \alpha\left[1 - u'(y^{\max})/\eta'(y^{\max})\right]$.

We will show that $V(\omega)$ has the same closed-form solution as derived in claim 1 when $\omega > \omega^{\max}$. Given this observation, we will establish that if $\omega = 0$ then $\omega' > \omega^{\max}$ and y can be solved in closed form.

Claim 2 *Suppose that $\hat{y} < y^{\max} < y^*$ and $\lambda < \alpha\left[1 - u'(y^{\max})/\eta'(y^{\max})\right]$. Then, there exists a unique (y_0, y_1) with $\hat{y} < y_1 < y^{\max} < y_0 < y^*$ that solves (14)-(15), and the unique V that solves (8)-(11) satisfies*

$$V(\omega) = \alpha[u(y_0) - v(y_0)] + \frac{\beta}{1-\beta}\alpha[u(y_1) - v(y_1)] \quad \text{if } \omega = 0, \tag{34}$$

$$= \frac{\alpha}{1-\beta}\{u[g(\omega)] - v[g(\omega)]\} \qquad \text{if } \omega \in [\omega^{\max}, \bar{\omega}], \tag{35}$$

where $g(\omega)$ is given in Part 1.

The fact that V satisfies (35) follows the proof of the second case in the claim in the proof of Part 1 and the Contraction Mapping Theorem. Note that by (35), $V'(\omega)$ is given by (33) for $\omega > \omega^{\max}$ and hence the proof there applies exactly.

Here we show (34). First we rewrite the problem in (8)-(11) at $\omega = 0$ as follows:

$$\max_{y,\omega'} \{\alpha\left[u(y) - v(y)\right] + \beta V(\omega')\} \tag{36}$$

$$\text{s.t.} \quad -\eta(y) + \beta\frac{\omega'}{\lambda} \geq 0 \tag{37}$$

$$y \in [0, y^*], \ \omega' \in [0, \bar{\omega}]. \tag{38}$$

Note that (10) is trivially satisfied when $\omega = 0$. Now, conjecturing that $\omega' \geq \omega^{\max}$, we can replace $V(\omega')$ by the expression given by (35), y by y_0 and $g(\omega')$ by y_1, and transform the above problem to

$$\max_{(y_0,y_1)\in[0,y^*]\times[\hat{y},y^{\max}]} \left\{\alpha\left[u(y_0) - v(y_0)\right] + \alpha\frac{u(y_1) - v(y_1)}{r}\right\} \tag{39}$$

$$\text{s.t.} \qquad -\eta(y_0) + \alpha\frac{u(y_1) - \eta(y_1)}{\lambda r} \geq 0, \tag{40}$$

which is exactly the same as (14)-(15). By the Kuhn-Tucker conditions, a pair (y_0, y_1) solves the above problem if it satisfies the following FOC and feasibility condition:

$$\frac{u'(y_0) - v'(y_0)}{\eta'(y_0)} = -\frac{\lambda}{\alpha}\left[\frac{u'(y_1) - v'(y_1)}{u'(y_1) - \eta'(y_1)}\right] \tag{41}$$

$$\alpha[u(y_1) - \eta(y_1)] = r\lambda\eta(y_0). \tag{42}$$

In order to show that the solution (y_0, y_1) is also a solution to the problem in (8)-(11) at $\omega = 0$ we only need to verify our conjecture,

$$\omega_1 = \frac{1}{1-\beta}\alpha[u(y_1) - \eta(y_1)] > \omega^{\max},$$

because the necessary conditions, (41)-(42), are also sufficient by the concavity of V over its entire domain.

Now we show that there exists a unique pair (y_0, y_1) with $\hat{y} < y_1 < y^{\max} < y_0 < y^*$ that satisfies (41)-(42). For each $y_1 \in (\hat{y}, y^{\max}]$, define

$$h(y_1) = \eta^{-1}\left[\frac{\alpha}{r\lambda}[u(y_1) - \eta(y_1)]\right].$$

as the unique solution of y_0 to (42) for a given y_1. Note that $h(y^{\max}) = y^{\max}$. For any $y_1 \in (\hat{y}, y^{\max}]$,

$$h'(y_1) = \frac{\alpha}{r\lambda}\frac{[u'(y_1) - \eta'(y_1)]}{\eta'[h(y_1)]} < 0.$$

Substituting y_0 by its expression given by $h(y_1)$ in the left side of (41), we rewrite (41) as $H(y_1) = 0$ where

$$H(y_1) = \frac{u'[h(y_1)] - v'[h(y_1)]}{\eta'[h(y_1)]} + \frac{\lambda}{\alpha}\left[\frac{u'(y_1) - v'(y_1)}{u'(y_1) - \eta'(y_1)}\right].$$

The function $H(y_1)$ is continuous and strictly increasing in $(\hat{y}, y^{\max}]$ with

$$\lim_{y_1 \downarrow \hat{y}} H(y_1) = -\infty,$$

and, at $y_1 = y^{\max}$, we have

$$\begin{aligned} H(y^{\max}) &= \frac{u'(y^{\max}) - v'(y^{\max})}{\eta'(y^{\max})} + \frac{\lambda}{\alpha}\left[\frac{u'(y^{\max}) - v'(y^{\max})}{u'(y^{\max}) - \eta'(y^{\max})}\right] \\ &= [u'(y^{\max}) - v'(y^{\max})]\left\{\frac{1}{\eta'(y^{\max})} + \frac{\lambda}{\alpha}\left[\frac{1}{u'(y^{\max}) - \eta'(y^{\max})}\right]\right\} > 0 \end{aligned}$$

because $\lambda < \alpha\,[1 - u'(y^{\max})/\eta'(y^{\max})]$. Thus, by Intermediate Value Theorem, there exists a unique $y_1 \in (\hat{y}, y^{\max})$ such that $H(y_1) = 0$ and hence (41) holds for $(h(y_1), y_1)$, and $h(y_1) > y^{\max}$ as h is strictly decreasing with $h(y^{\max}) = y^{\max}$. This proves that there exists a unique pair (y_0, y_1) with $\hat{y} < y_1 < y^{\max} < y_0 < y^*$ that satisfies (41) and (42).

Finally, because $\hat{y} < y_1 < y^{\max} < y_0 < y^*$ and because $(\omega', y) = (\omega_1, y_0)$ with $\omega_1 = \alpha[u(y_1) - \eta(y_1)]/(1-\beta)$ is the solution to the maximization problem in (8)-(11) for $\omega = 0$, V satisfies (34).