Lending Relationships and Optimal Monetary Policy*

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Abstract

We construct and calibrate a monetary model of corporate finance with endogenous formation of lending relationships. The equilibrium features money demands by firms that depend on their access to credit and a pecking order of financing means. We describe the mechanism through which monetary policy affects the creation of relationships and firms' incentives to use internal or external finance. We study optimal monetary policy following an unanticipated destruction of relationships under different commitment assumptions. The Ramsey solution uses forward guidance to expedite creation of new relationships by committing to raise the user cost of cash gradually above its long-run value. Absent commitment, the user cost is kept low, delaying recovery.

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1 Introduction

Most businesses, especially small ones, rely on secure access to credit through stable relationships with banks. According to the 2003 Survey of Small Business Finances, 65% of small businesses that need funding had access to a credit line or revolving credit arrangement – a proxy for lending relationships. These firms hold 30% less cash relative to firms not in a lending relationship, thereby suggesting some degree of substitutability between internal finance with cash and external finance through banking relationships. Insofar as monetary policy affects the user cost of cash, these observations suggest an asymmetric transmission of monetary policy to firms depending on their access to lending relationships.

Monetary policy transmission to relationship lending is especially critical in times of financial crisis as a fraction of these relationships get destroyed due to bank failures, stricter application of loan covenants, or tighter lending standards.[1] During the Great Depression, the destruction of lending relationships explained one-eighth of the economic contraction (Cohen, Hachem, and Richardson, 2021). The goal of this paper is to understand the mechanism through which monetary policy affects the creation of lending relationships, financing of firms, and the policymaker's trade-offs in times of crisis.

We develop a general equilibrium model of lending relationships and corporate finance in the tradition of the New Monetarist approach (surveyed in Lagos, Rocheteau, and Wright, 2017) and use it to study optimal monetary policy, with and without commitment, in the aftermath of a crisis. In the model economy, entrepreneurs receive idiosyncratic investment opportunities, as in Kiyotaki and Moore (2005), which can be financed with bank credit or retained earnings in liquid assets. The rate of return of liquid assets, and hence the interest rate spread between liquid and illiquid assets, is controlled by the monetary authority. We assume the frequency of investment opportunities that can be financed expands when a firm enters a relationship with a bank. Hence, external finance through banks plays an essential role, even when the interest rate spread goes to zero (which corresponds to the Friedman rule). The role of banks consists of issuing IOUs that are acceptable means of payment (inside money) in exchange for the illiquid IOUs of the entrepreneurs with whom they have a relationship. However, in the presence of search and information frictions,

[1] During the Great Recession, the number of small business loans contracted by a quarter from its peak (FFIEC Call Reports; Chen, Hanson, and Stein, 2017), and distressed banks reneged on precommitted, formal lines of credit (Huang, 2009).

relationships take time to form and are costly to monitor.

The transmission of monetary policy operates through two distinct channels. There is a liquidity channel where a fall in the rate of return of liquid assets raises the interest spread between liquid and illiquid assets and decreases holdings of liquidity for all firms. Consistent with the evidence (see, e.g., Section 4.2), this effect is asymmetric across firms with different access to credit. Under fairly general conditions, firms in a lending relationship hold less liquid assets than unbanked firms, and this gap widens as the interest spread between liquid and illiquid assets increases. Banked firms respond more strongly to an increase in the user cost of liquid assets than unbanked firms by substituting away from internal to external finance.

Second, there is a novel lending channel operating through the creation of relationships. An increase in the interest spread between liquid and illiquid assets makes it more profitable for a firm to be in a banking relationship. Indeed, relationship lending allows firms to economize on their holdings of liquid assets, and the associated cost saving increases as liquidity becomes more expensive. Critically, since banks have some bargaining power, they can raise the revenue they collect from firms through higher interest payments or fees, which gives them an incentive to create more relationships.

We put our model to work by investigating the economy's response to a negative credit shock described as an exogenous and unanticipated destruction of lending relationships starting from steady state. Under a policy rule that keeps the supply of liquid assets constant, the interest spread jumps up initially, thereby stimulating the creation of new relationships, before gradually declining to its initial level. In contrast, if the supply of liquid assets is perfectly elastic, aggregate liquidity increases while the rate of credit creation remains constant. In a calibrated version of the model, the policy that consists of keeping the supply of liquidity constant generates a decline in aggregate investment which is twice as large as the one obtained under a constant interest rate, but the recovery in terms of lending relationships is faster.

We then turn to studying the optimal monetary policy response under different assumptions regarding the commitment power of the policymaker. If the policymaker can commit, optimal policy entails setting low spreads close to the zero lower bound at the outset of the crisis to promote internal finance by newly unbanked firms. To maintain banks' incentives to participate in the market for relationships despite low interest spreads, the policymaker uses "forward guidance" by promising high spreads in the future. If the shock is sufficiently large, the time path of spreads is hump-shaped, i.e., spreads in the medium run overshoot their long run value.

If the policymaker cannot commit and sets the interest spread period by period, then the optimal policy maintains banks' incentives to create lending relationships by raising spreads in the short run since it cannot commit to raising them in the future. The larger the contraction, the higher the policymaker raises spreads in response. We find the recovery is slower than under commitment; e.g., the half life of the transition path to steady state following a 60% contraction, absent commitment, is 26 months in our calibrated example, compared with 23 months under the Ramsey policy. This produces a welfare loss from lack of commitment ranging from 0.25% to 0.35% of consumption, depending on the size of the shock.

Literature

There are four main approaches to the role of lending relationships: insurance in the presence of uncertain investment projects (Berlin and Mester, 1999), monitoring in the presence of agency problems (Holmstrom and Tirole, 1997), screening with hidden types (Agarwal and Hauswald, 2010), and dynamic learning under adverse selection (Sharpe, 1990; Hachem, 2011).[2] We adopt the insurance approach as the need for insurance in the presence of idiosyncratic shocks is what generates both the demand for liquid assets (money) and the usefulness of banks, and it is this role that is central to the monetary policy trade-off we are focusing on. The screening of borrowers is captured by search and matching frictions in the credit market (as in, e.g., Wasmer and Weil, 2004).[3] The monitoring role of banks is captured by a real resource cost from providing loans and is similar in spirit as Diamond (1984) and Holmstrom and Tirole (1997).

There is a small literature on monetary policy and relationship lending, e.g., Hachem (2011) and Bolton, Freixas, Gambacorta, and Mistrulli (2016).[4] Our model differs from that literature by emphasizing money demand by firms and their choice between internal and external finance, endogenizing the creation of relationships through a frictional matching technology, and assuming banks have bargaining power. In particular, the endogenous choice of firms' money holdings is

[2] See von Thadden (2004) for a discussion of Sharpe (1990). See Elyasiani and Goldberg (2004) and references therein for a survey of the corporate finance literature on relationship lending.

[3] Modeling screening as a search process is conceptually distinct from modeling it as a mechanism design problem under private information. Models that combine search and adverse selection include Inderst (2001) and Guerrieri, Shimer, and Wright (2010). In the context of banking, Hachem (2021) distinguishes explicitly the matching of banks with borrowers from the screening of borrowers' types where both activities require some effort.

[4] Bolton and Freixas (2006) is a related paper that focuses on monetary policy and transaction lending. Boualam (2017) models relationship lending with directed search and agency costs but does not have an endogenous demand for liquid assets or monetary policy.

key to explain the transmission mechanism of monetary policy and raises issues, such as the co-essentiality of money and bank credit, that do not arise in traditional corporate finance models. This choice is related to the formalization of liquidity demand in Holmstrom and Tirole (2011, Ch. 2), except we formalize repeated investment opportunities in an infinite horizon rather than a two-period model.

Our description of the credit market with search frictions is analogous to den Haan, Ramey, and Watson (2003), Wasmer and Weil (2004), Petrosky-Nadeau and Wasmer (2017), and models of OTC dealer markets by Duffie, Garleanu, and Pedersen (2005) and Lagos and Rocheteau (2009). Drechsler, Savov, and Schnabl (2017) document empirically the importance of bank market power for the transmission of monetary policy, but focus on the deposit market while our focus is on bank lending. We view their approach as complementary to ours.[5]

Our model is a corporate finance version of Lagos and Wright (2005) and its competitive version by Rocheteau and Wright (2005). The closest paper is Rocheteau, Wright, and Zhang (2018) which studies monetary policy transmission with transaction lenders when firms are subject to pledgeability constraints. Imhof, Monnet, and Zhang (2018) extend the model by introducing limited commitment by banks and risky loans.[6] Our formalization of banks is similar to the one in Gu, Mattesini, Monnet, and Wright (2016) and references therein. Models of money and credit with long-term relationships include Corbae and Ritter (2004) with indivisible money and Rocheteau and Nosal (2017, Ch. 8) with divisible money. Our description of a crisis is analogous to the one in Weill (2007) and Lagos, Rocheteau, and Weill (2011).

Our recursive formulation of the Ramsey problem is related to Chang (1998) and Aruoba and Chugh (2010) in the context of the Lagos-Wright model. Our approach to the policy problem without commitment is similar to Klein, Krusell, and Ríos-Rull (2008) and Martin (2011, 2013) in a New Monetarist model where the government finances the provision of a public good with money, nominal bonds, and distortionary taxes. Unlike the usual perturbation method applying to the steady state, we devise an algorithm based on contraction mappings to compute transitional dynamics.

[5] Choi and Rocheteau (2021) formalize the bank deposit market following a similar methodology as in our paper and show that it generates a deposits channel as documented in Drechsler, Savov, and Schnabl (2017).

[6] Here, a lending relationship is a commitment by the bank to provide firms with conditional access to credit. While in models building on Kehoe and Levine (1993) or Alvarez and Jermann (2000), limited commitment generates an endogenous debt limit, in our model, it is costly enforcement that puts a limit on banks' willingness to lend. See Raveendranathan (2020) for a model of revolving credit lines where a credit contract specifies an interest rate and credit limit.

2 Environment

Time is indexed by $t \in \mathbb{N}_0$. Each period is divided in three stages. In the first stage, a competitive market for capital goods opens and investment opportunities arise. The second stage is a frictional market where long-term lending relationships are formed. The last stage is a frictionless centralized market where agents trade assets and consumption goods and settle debts. Figure 1 summarizes the timing of a representative period.

There are two goods: a capital good, k, storable across stages but not across periods, and a consumption good, c, taken as the numéraire. There are three types of agents: *entrepreneurs* that need capital, *suppliers* that can produce capital, and *banks* that can finance the acquisition of capital as explained below. The population of entrepreneurs is normalized to one. Given CRS for the production of capital goods (see below), the population size of suppliers is immaterial. The population of active banks is endogenous and will be determined through free entry. All agents have linear preferences, $c - h$, where c is consumption of numéraire, and h is labor. They discount across periods according to $\beta = 1/(1+\rho)$, where $\rho > 0$.

In stage 1, entrepreneurs have probabilistic access to a technology that transforms k units of capital goods into $y(k)$ units of numéraire in stage 3. We assume $y(k)$ is continuously differentiable with $y' > 0$, $y'' < 0$, $y'(0) = +\infty$, and $y'(+\infty) = 0$. Production/investment opportunities are i.i.d across time and entrepreneurs: they occur with probability λ^u for unbanked entrepreneurs and $\lambda^b \geq \lambda^u$ for banked entrepreneurs. In Section 6, we endogenize λ^b and λ^u as the outcome of a costly search/screening for investment opportunities by entrepreneurs and banks and show that it suffices that those efforts are not perfect substitutes to generate $\lambda^b > \lambda^u$. Capital k is produced by suppliers in stage 1 with a linear technology, $k = h$. Social efficiency dictates $k = k^*$ where $y'(k^*) = 1$. Agents can also produce c using their labor in stage 3 with a linear technology, $c = h$.

Entrepreneurs lack commitment, have private trading histories, and do not interact repeatedly with the same suppliers. As a result, suppliers do not accept IOUs issued by entrepreneurs who have no consequences to fear from reneging. In contrast, banks have access to a commitment technology that allows them to issue liabilities that are repaid in the last stage. Banks also have the technology to enforce the repayment of entrepreneurs' IOUs.[7] These technologies are operated at a cost $\psi(\mathrm{L})$, where L is both the liabilities (in terms of the numéraire) issued by the bank to be repaid in stage

[7] We endogenize bank commitment through reputation in the appendix of Rocheteau, Wright, and Zhang (2018). The possibility of insolvent banks in a version of our model is studied by Imhof, Monnet, and Zhang (2018).

3 and the principal on the entrepreneur's loan. This cost is increasing and convex, i.e., $\psi'(\text{L}) > 0$, $\psi''(\text{L}) > 0$, $\psi(0) = \psi'(0) = 0$. It includes the costs to issue liabilities that are easily recognizable and noncounterfeitable, costs associated with the commitment to repay L, and costs to monitor the entrepreneurs' loans backing these liabilities. We define $\bar{\text{L}} \equiv \arg\max\{y(\text{L}) - \text{L} - \psi(\text{L})\} \leq k^*$ as the natural limit on loan sizes imposed by monitoring costs.

To be eligible for external finance, an entrepreneur must form a lending relationship with a bank. Each bank manages at most one relationship.[8] These relationships are formed in stage 2. At the beginning of stage 2, banks without a lending relationship decide whether to participate in the credit market at a disutility cost, $\zeta > 0$. There is then a bilateral matching process between unbanked entrepreneurs and unmatched banks. The number of new relationships formed in stage 2 of period t is $\alpha_t = \alpha(\theta_t)$, where θ_t is the ratio of unmatched banks to unbanked entrepreneurs, defined as credit market tightness. We assume $\alpha(\theta)$ is increasing and concave, $\alpha(0) = 0$, $\alpha'(0) = 1$, $\alpha(\infty) = 1$, and $\alpha'(\infty) = 0$. Since matches are formed at random, the probability an entrepreneur matches with a bank is α_t, and the probability a bank matches with an entrepreneur is $\alpha_t^b = \alpha(\theta_t)/\theta_t$. In our context, search frictions can be interpreted as the time it takes to gather information about firms and to match heterogenous investment opportunities to banks with different expertise. We denote the elasticity of the matching function $\epsilon(\theta) \equiv \alpha'(\theta)\theta/\alpha(\theta)$. A match existing for more than one period is terminated at the end of the second stage with probability $\delta \in (0, 1)$. Newly formed matches are not subject to the risk of termination.

In addition to banks' short-term liabilities, there are risk-free assets which are storable across periods and promise a real rate of return from period t to $t+1$ equal to r_{t+1}. We assume r_{t+1} is set by the policymaker; e.g., the policymaker determines the supply of money or government bonds.[9] If liquid assets take the form of fiat money, then r_{t+1} is approximately equal to the opposite of the inflation rate and is implemented through changes in the money growth rate. If $r_{t+1} = \rho$, the cost of holding liquid assets is zero, which is interpreted as the Friedman rule. In a large class of monetary models, the Friedman rule allows agents to perfectly self-insure against idiosyncratic shocks, thereby eliminating a role for credit or banks. When we turn to the optimal monetary policy, we make use of the assumption $\lambda^b > \lambda^u$ to maintain a role for banks even at the Friedman rule.[10] In that case,

[8] This assumption is analogous to the Pissarides (2000) one-firm-one-job assumption. One can think of actual banks as a large collection of such relationships.

[9] Implicitly, changes in government liabilities (i.e., money or bonds) are implemented with lump sum transfers or taxes.

[10] We think of λ^b/λ^u as a generic wedge between self-insurance with liquid assets and insurance through a lending

a fraction of investment opportunities cannot be financed without a relationship with a bank.

3 Liquidity and lending relationships

We study equilibria where investment opportunities are financed with bank loans and liquid assets accumulated from retained earnings.

3.1 Value functions

Notations for value functions of entrepreneurs and banks in different states and different stages are summarized in Figure 1. We characterize these value functions from stage 3 and move backward to stage 1.

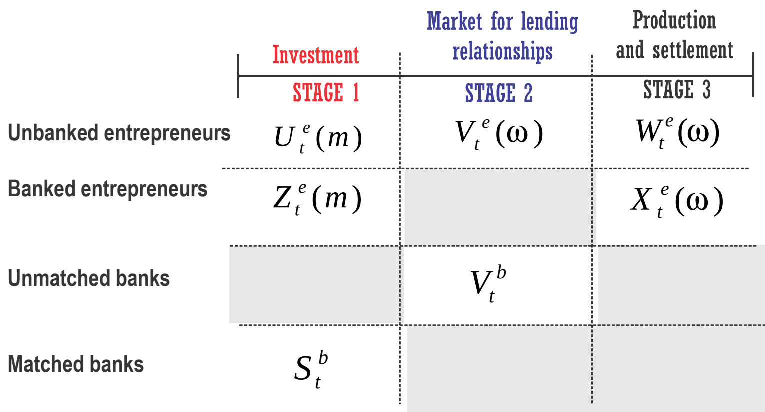

Figure 1: Timing of a representative period and value functions

STAGE 3 (Settlement and portfolio choices). The lifetime expected utility of an unbanked entrepreneur with wealth ω (expressed in terms of numéraire) in the last stage of period t is

$$W_t^e(\omega) = \max_{c_t, h_t, m_{t+1} \geq 0} \left\{ c_t - h_t + \beta U_{t+1}^e(m_{t+1}) \right\} \quad \text{s.t.} \quad m_{t+1} = (1 + r_{t+1})(\omega + h_t - c_t),$$

where $U_t^e(m)$ is the value function of an unbanked entrepreneur at the beginning (stage 1) of period t with liquid wealth m. The entrepreneur saves $\omega + h_t - c_t$ from his current wealth and income in the form of liquid assets. The rate of return on liquid assets is r_{t+1}, hence, holdings in period

relationship. Our leading interpretation is that banks, through their monitoring activities, can generate information about profitable investment opportunities for their customers. We formalize this idea in Section 6. There are alternative reasons discussed in the literature for why entrepreneurs cannot perfectly self insure. For instance, liquid assets might not be perfectly acceptable by suppliers, as in Lester, Postlewaite, and Wright (2012). They might be subject to theft or embezzlement, as in Sanches and Williamson (2010), or can be diverted away from investment as in Holmstrom and Tirole (2011, Ch. 2).

$t+1$ are $m_{t+1} = (1 + r_{t+1})(\omega + h_t - c_t)$. Substituting $c_t - h_t$ from the budget identity into the objective, the Bellman equation becomes

$$W_t^e(\omega) = \omega + \max_{m_{t+1} \geq 0} \left\{ -\frac{m_{t+1}}{1 + r_{t+1}} + \beta U_{t+1}^e(m_{t+1}) \right\}. \tag{1}$$

As is standard in models with risk-neutral agents, value functions are linear in wealth, and the choice of m_{t+1} is independent of ω. By a similar reasoning, the lifetime utility of a banked entrepreneur with wealth ω in the last stage of period t, $X_t^e(\omega)$, solves

$$X_t^e(\omega) = \omega - \frac{m_{t+1}^b}{1 + r_{t+1}} + \beta Z_{t+1}^e(m_{t+1}^b), \tag{2}$$

where $Z_t^e(m)$ is the value of a banked entrepreneur at the beginning of period t with m units of liquid assets, and m_{t+1}^b is the amount of liquid assets that the banked entrepreneur must hold as specified by the lending-relationship contract.

STAGE 2 (Market for lending relationships). The lifetime expected utility of an unbanked entrepreneur at the beginning of the second stage solves

$$V_t^e(\omega) = \alpha_t X_t^e(\omega) + (1 - \alpha_t) W_t^e(\omega) = \omega + \alpha_t X_t^e(0) + (1 - \alpha_t) W_t^e(0). \tag{3}$$

With probability α_t, the unmatched entrepreneur enters a lending relationship and, with probability $1 - \alpha_t$, he proceeds to the last stage unmatched. From the right side of (3), $V_t^e(\omega)$ is linear in ω.

The lifetime discounted profits of a bank entering at time t, V_t^b, solve

$$V_t^b = -\zeta + \alpha_t^b \beta \mathcal{S}_{t+1}^b + \left(1 - \alpha_t^b\right) \beta \max\left\{V_{t+1}^b, 0\right\}. \tag{4}$$

From (4), an unmatched bank incurs a cost ζ at the start of the second stage to participate in the credit market; there, the bank is matched with an entrepreneur with probability $\alpha_t^b = \alpha(\theta_t)/\theta_t$ and remains unmatched with probability $1 - \alpha_t^b$. The discounted sum of the profits from a lending relationship is $\mathcal{S}_{t+1}^b$.

STAGE 1 (Investment opportunities). In the first stage, suppliers choose the amount of k to produce at a linear cost taking its price in terms of numéraire, q_t, as given. Formally, they solve $\max_{k \geq 0} \{-k + q_t k\}$. If the capital market is active, $q_t = 1$. The lifetime utility of an unbanked entrepreneur at the beginning of period t is

$$U_t^e(m_t) = \mathbb{E}\left[V_t^e(\omega_t)\right] \quad \text{s.t.} \quad \omega_t = m_t + \chi_t^u \max_{k_t \leq m_t} \left[y(k_t) - k_t\right], \tag{5}$$

where χ_t^u is a Bernoulli variable equal to one with probability λ^u if the entrepreneur receives an investment opportunity. The entrepreneur's wealth when entering the second stage, ω_t, consists of his initial wealth, m_t, and profits from the investment opportunity if $\chi_t^u = 1$. To maximize profits, the entrepreneur chooses k_t subject to the liquidity constraint $k_t \leq m_t$. By the linearity of $V_t^e(\omega_t)$, (5) reduces to

$$U_t^e(m_t) = \lambda^u \max_{k_t \leq m_t} [y(k_t) - k_t] + m_t + V_t^e(0). \tag{6}$$

The lifetime expected utility of a banked entrepreneur with m liquid assets at the beginning of stage 1 solves

$$Z_t^e(m) = \mathbb{E}[\delta W_t^e(\omega_t) + (1-\delta)X_t^e(\omega_t)] \quad \text{s.t.} \quad \omega_t = m + \chi_t^b \left[y(k_t^b) - k_t^b\right] - \phi_t, \tag{7}$$

where k_t^b is the investment level, and ϕ_t is an intermediation fee due in stage 3. The indicator variable, χ_t^b, equals one if the banked entrepreneur receives an investment opportunity with probability λ^b. The quantities, (k_t^b, ϕ_t), are determined as part of an optimal contract. Using the linearity of $W_t^e(\omega_t)$ and $X_t^e(\omega_t)$,

$$Z_t^e(m) = \mathbb{E}[\omega_t] + \delta W_t^e(0) + (1-\delta)X_t^e(0), \tag{8}$$

where the entrepreneur's expected wealth at the end of a period is

$$\mathbb{E}[\omega_t] = m - \phi_t + \lambda^b \left[y(k_t^b) - k_t^b\right].$$

From (8), the lending relationship is destroyed with probability δ, in which case the entrepreneur's value in the last stage is W_t^e. Otherwise, the continuation value is X_t^e.

Finally, the discounted sum of bank profits from a lending relationship at the start of period t, $\mathcal{S}_t^b$, solves

$$\mathcal{S}_t^b = \phi_t - \lambda^b \psi\left(k_t^b - m_t^b\right) + \beta(1-\delta)S_{t+1}^b, \tag{9}$$

where m_t^b is the holdings of liquid assets of the entrepreneur matched with the bank to be used as down payment for a loan. The second term on the right side is the cost of the loan $\mathrm{L}_t = k_t^b - m_t^b$.

3.2 Optimal liquidity of unbanked entrepreneurs

We now determine the optimal holdings of liquid assets by unbanked entrepreneurs. Substituting $U_t^e(m_t)$ from (6) into (1), an unbanked entrepreneur's choice of liquid assets is a solution to

$$\pi_t^u = \pi^u(s_t) \equiv \max_{m_t \geq 0} \left\{-s_t m_t + \lambda^u \max_{k_t \leq m_t} [y(k_t) - k_t]\right\}, \tag{10}$$

where the interest rate spread between liquid and illiquid asset is

$$s_t \equiv \frac{\rho - r_t}{1 + r_t}. \tag{11}$$

If the liquid asset does not bear interest (e.g., cash), then s is the nominal rate on an illiquid bond, and its lower bound is zero.[11] The FOC associated with (10) is

$$s_t = \lambda^u \left[y'(m_t^u) - 1 \right], \tag{12}$$

where m_t^u denotes the demand for liquid assets by unbanked entrepreneurs. The term on the left side is the cost of holding liquid assets, whereas the right side is the expected marginal benefit from holding an additional unit of the liquid asset, which is the probability of an investment opportunity times the marginal profits from an additional unit of capital. The optimal liquid wealth of an unbanked entrepreneur decreases with s_t but increases with λ^u.

3.3 Optimal lending relationship contract

The lending relationship contract negotiated in stage 2 of period $t-1$ between newly matched entrepreneurs and banks is a list, $\{\phi_{t+\tau}, k_{t+\tau}^b, m_{t+\tau}^b\}_{\tau=0}^{\infty}$, where $\phi_{t+\tau}$ is the fee to the bank, $k_{t+\tau}^b$ is the investment level, and $m_{t+\tau}^b$ is the amount of liquid wealth to be used as down payment on loans. So, the loan size is $\mathrm{L}_{t+\tau} = k_{t+\tau}^b - m_{t+\tau}^b$.

The entrepreneur's surplus from being in a lending relationship in the third stage of $t-1$ is defined as $\mathcal{S}_t^e = \left[X_{t-1}^e(0) - W_{t-1}^e(0) \right] / \beta$. The bank's surplus is $\mathcal{S}_t^b$. The terms of the lending relationship contract are chosen to maximize the generalized Nash product, $\left(\mathcal{S}_t^b\right)^{\eta} \left(\mathcal{S}_t^e\right)^{1-\eta}$, where η is the bargaining power of banks. As is standard in bargaining problems with transferable utilities, $\{k_{t+\tau}^b, m_{t+\tau}^b\}_{\tau=0}^{\infty}$ is chosen to maximize the total surplus of a lending relationship, $\mathcal{S}_t = \mathcal{S}_t^e + \mathcal{S}_t^b$, while $\{\phi_{t+\tau}\}_{\tau=0}^{\infty}$ splits the surplus according to each party's bargaining power. In the proof of Proposition 1, we show the total surplus of a relationship solves

$$\mathcal{S}_t = \lambda^b \left[y(k_t^b) - k_t^b - \psi(k_t^b - m_t^b) \right] - s_t m_t^b - \underline{\pi}_t + (1-\delta)\beta \mathcal{S}_{t+1}, \tag{13}$$

where

$$\underline{\pi}_t = \pi_t^u + V_t^e(0) - W_t^e(0) \tag{14}$$

represents the opportunity cost of being in a lending relationship and is composed of the expected profits of an unbanked entrepreneur (net of the cost of holding liquid wealth), π_t^u, augmented with

[11] This expression for the cost of holding liquid assets as a spread is consistent with the construction in Barnett (1980).

the value of the matching opportunities in the credit market, $V_t^e(0) - W_t^e(0)$. The first term on the right side of (13) corresponds to the profits of an investment opportunity financed both internally with m_t^b units of liquid wealth and externally with a loan of size $\text{L}_t = k_t^b - m_t^b$. The second term is the entrepreneur's cost of holding m_t^b units of liquid wealth.

Proposition 1 ***(Optimal lending contract with internal finance).*** *The terms of the optimal lending relationship contract solve*

$$\psi'\left(k_t^b - m_t^b\right) = y'(k_t^b) - 1 \leq \frac{s_t}{\lambda^b}, \quad \text{"="} \text{ if } m_t^b > 0, \quad \forall t. \tag{15}$$

The intermediation fee is equal to

$$\phi_t = \lambda^b \psi(k_t^b - m_t^b) + \eta\left[\pi^b(s_t) - \pi^u(s_t)\right] - (1-\eta)\zeta\theta_t, \tag{16}$$

where the joint expected profits net of the cost of holding assets are

$$\pi^b(s_t) = \max_{k^b, m^b \geq 0}\left\{\lambda^b\left[y(k^b) - k^b - \psi(k^b - m^b)\right] - s_t m^b\right\}. \tag{17}$$

The optimal contract is consistent with a pecking order where firms fund investment projects with internal finance first and resort to external finance last. Indeed, conditional on an entrepreneur holding m^b units of liquid assets, the optimal loan contract solves

$$\varpi(m^b) = \max_{k^b, \text{L}}\left\{y(k^b) - k^b - \psi(\text{L})\right\} \quad \text{s.t.} \quad k^b \leq \text{L} + m^b. \tag{18}$$

If $m^b \geq k^*$, the entrepreneur finances all of the project internally, $k^b = k^*$ and $\text{L}= 0$. If $m^b < k^*$, the entrepreneur finances m^b units of capital internally and L units externally, where L is chosen to equalize the net marginal return of capital, $y'(k_t^b) - 1$, and the marginal cost of external finance, $\psi'(\text{L}_t)$. Given $\varpi(m^b)$, the optimal holdings of liquid assets of a banked entrepreneur solve

$$m^b = \arg\max_{m^b \geq 0}\left\{\lambda^b \varpi(m^b) - s m^b\right\}. \tag{19}$$

Using $\varpi'(m^b) = \psi'(\text{L})$, the first-order condition for the optimal holdings of liquid assets satisfies (15). The inequality in (15) states that the marginal gain from financing investment internally, $\lambda^b \psi'\left(k_t^b - m_t^b\right)$, cannot be greater than the opportunity cost of holding liquid assets, s_t.

The intermediation fee in (16) consists of the average cost of monitoring loans, a fraction η of the entrepreneur's profits from being in a lending relationship, net of a fraction $1-\eta$ of the bank's entry costs. It depends on s_t through the term $\Delta\pi(s_t) \equiv \pi^b(s_t) - \pi^u(s_t)$, where $\partial\Delta\pi(s_t)/\partial s_t = m_t^u - m_t^b$.

The difference in liquid wealth between unbanked and banked entrepreneurs provides a channel through which policy can affect bank profits and their incentive to participate in the credit market. For instance, if $m_t^u > m_t^b$, then an increase in s_t raises banks' profits.

Since banks' only interest-earning assets are the loans provided to entrepreneurs, and their liabilities do not bear interest, we identify the average net interest margin on a lending relationship as

$$NIM_t = \frac{\phi_t}{\lambda^b \left(k_t^b - m_t^b\right)}. \tag{20}$$

The numerator is the fees paid by banked entrepreneurs, while the denominator is the sum of all loans.

While from Proposition 1, the terms of the optimal lending relationship contract do not depend directly on the survival probability of the relationship, captured by $1-\delta$, the joint surplus, $\mathcal{S}_t$, defined in (13) increases with $1-\delta$.[12] Relationships are more valuable as they last longer. But since relationships are costly to form, their expected duration matters in general equilibrium. This effect will be captured by the last term of (16) once we endogenize market tightness.

3.4 Creation of lending relationships

Free entry of banks in the market for relationship lending means $V_{t+1}^b \leq 0$, with equality if there is entry.

Lemma 1 *In any equilibrium where the market for relationship lending is active in all periods,* $\{\theta_t\}_{t=0}^{\infty}$*, solves*

$$\frac{\theta_t}{\alpha(\theta_t)} = \frac{\beta\eta\Delta\pi(s_{t+1})}{\zeta} - \beta(1-\eta)\theta_{t+1} + \beta(1-\delta)\frac{\theta_{t+1}}{\alpha(\theta_{t+1})}. \tag{21}$$

According to (21), monetary policy affects the creation of lending relationships through the term $\Delta\pi(s_{t+1})$. If $m_{t+1}^u > m_{t+1}^b$, e.g., if $\lambda^u = \lambda^b$ and $s_{t+1} > 0$, then an increase in s_{t+1} raises $\Delta\pi(s_{t+1})$ by reducing the net profits of unbanked entrepreneurs by more than the profits of banked entrepreneurs. This effect worsens an entrepreneur's status quo in the negotiation and raises ϕ_t. As a result, bank profits increase with s_{t+1}. If $m_{t+1}^u < m_{t+1}^b$, the opposite is true and bank profits decrease with s_{t+1}.

The measure of lending relationships at the start of a period evolves according to

$$\ell_{t+1} = (1-\delta)\ell_t + \alpha_t(1-\ell_t). \tag{22}$$

[12] We could obtain a direct effect of $1-\delta$ on the terms of the lending relationship contract by changing the timing of the destruction of relationships, e.g., by assuming that relationships are destroyed at the beginning of a period.

The number of lending relationships at the beginning of $t+1$ equals the measure of lending relationships at the beginning of t that have not been severed, $(1-\delta)\ell_t$, plus newly created relationships, $\alpha_t(1-\ell_t)$.

3.5 Equilibrium

An equilibrium with internal and external finance is a bounded sequence, $\{\theta_t, \ell_t, m_t^u, m_t^b, k_t^b, \phi_t\}_{t=0}^{\infty}$, that solves (12), (15), (16), (21), and (22) for a given $\ell_0 > 0$. In the following proposition we characterize equilibria when banked and unbanked entrepreneurs receive investment opportunities at the same frequency.

Proposition 2 ***(Equilibria with internal and external finance).*** *Suppose $\lambda^u = \lambda^b = \lambda$. A unique steady-state monetary equilibrium exists and features an active credit market if and only if*

$$(\rho + \delta)\,\zeta < \eta \Delta\pi(s). \tag{23}$$

Let $\hat{k} > 0$ denote the solution to $y'(\hat{k}) = 1 + \psi'\left(\hat{k}\right)$. There are two regimes.

1. Low spread regime: $s_t \leq \hat{s} \equiv \lambda\psi'\left(\hat{k}\right)$. All entrepreneurs invest k_t that solves (12). The difference in their asset holdings according to their banking status is

$$m_t^u - m_t^b = \psi'^{-1}\left(\frac{s_t}{\lambda}\right). \tag{24}$$

In the neighborhood of $s = 0$, $\partial m^b/\partial s < \partial m^u/\partial s < 0$ and $\partial\theta/\partial s = 0$.

2. High spread regime: $s_t > \hat{s} \equiv \lambda\psi'\left(\hat{k}\right)$. Banked entrepreneurs hold no assets, $m_t^b = 0$, and invest $k_t^b = \hat{k}$.

Transmission of monetary policy. For all $s > 0$,

$$\frac{\partial\theta}{\partial s} = \eta\frac{\left(m^u - m^b\right)}{\zeta}\left\{\frac{(\rho+\delta)\left[1-\epsilon(\theta)\right]}{\alpha(\theta)} + 1 - \eta\right\}^{-1} > 0\ .$$

Proposition 2 distinguishes two regimes. If the cost of holding assets is low, between zero and $\hat{s}$, both banked and unbanked entrepreneurs invest the same amount. At one limit, when $s = 0$, internal finance is costless and all investment opportunities are financed internally, $m_t^u = m_t^b = k_t^b = k^*$. At the other limit, when $s = \hat{s}$, entrepreneurs invest $\hat{k}$, which is financed internally by unbanked entrepreneurs and externally by banked entrepreneurs. In between zero and $\hat{s}$, banked entrepreneurs make a down payment, $m_t^b < k_t$, and take a bank loan to cover the rest of their financing needs, while unbanked entrepreneurs cover all investment expenditures with their liquid assets, $m_t^u = k_t$. From (21), credit market tightness is determined by

$$\frac{(\rho+\delta)\theta\zeta}{\alpha(\theta)} = \lambda \mathrm{L}\left[NIM - \frac{\psi(\mathrm{L})}{\mathrm{L}}\right] \text{ with } NIM_t = \frac{\eta}{\lambda}s_t + \frac{(1-\eta)[\psi(\mathrm{L}_t) - \zeta\theta_t/\lambda]}{\mathrm{L}_t}.$$

The creation of relationships by banks depends on the volume of loans, L, the NIM, and the unit cost of external finance, $\psi(\mathrm{L})/\mathrm{L}$. The NIM is composed of two terms. The first term is proportional to the interest rate spread. The second term is a function of monitoring and entry costs.

In the second regime, when the cost of holding assets is larger than $\hat{s}$, banked entrepreneurs do not hold liquid wealth and resort to external finance only. If external finance is costless, $\psi \equiv 0$, then $\hat{s} = 0$ and only the second regime prevails. In that case, the equilibrium features $m_t^b = 0$ for all $s_t > 0$ and $k_t^b = \hat{k} = k^*$, i.e., investment levels are socially efficient. If external finance is costly, then $k^b > k^u$. We represent these two regimes in the left panel of Figure 2.

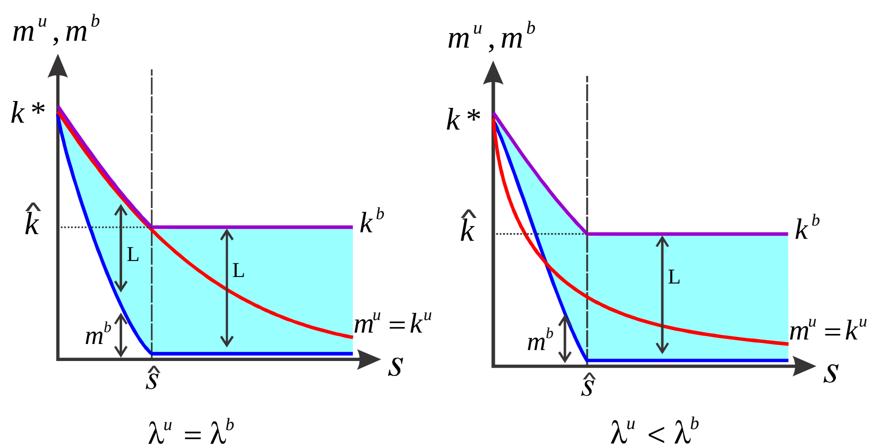

Figure 2: Holdings of liquid assets, loan sizes, and investment

We now turn to the case where $\lambda^b > \lambda^u$. There are still two regimes depending on whether s is smaller or larger than $\hat{s} = \lambda^b \psi'\left(\hat{k}\right)$. If $s < \hat{s}$, all entrepreneurs hold some liquidity, but the relationship between $m^u - m^b$ and s can be nonmonotone.

Proposition 3 ***(Money demands and banking status).*** *Suppose the cost of external finance is* $\psi(\mathrm{L}) = \psi_0 \mathrm{L}^{1+\xi}/(1+\xi)$ *and* $\lambda^b > \lambda^u$. *If* $\xi < 1$ *or* $\xi = 1$ *and* $\left(\lambda^b - \lambda^u\right)/\lambda^u > -y''(k^*)/\psi_0$, *then there exists* $0 < s_0 \leq s_1 \leq \hat{s}$ *such that for all* $s < s_0$, $m^b > m^u$ *and* $\partial\theta/\partial s < 0$*; for all* $s > s_1$, $m^u > m^b$ *and* $\partial\theta/\partial s > 0$.

For sake of illustration, suppose the cost of external finance is quadratic, $\xi = 1$, and ψ_0 is sufficiently large. When s is low ($s < s_0$), it is optimal for banked entrepreneurs who receive more frequent investment opportunities to accumulate more liquid assets than unbanked entrepreneurs. In contrast, if s is large ($s > s_1$), banked entrepreneurs can rely on bank credit to finance investment, and hence they hold less liquid assets than unbanked entrepreneurs. The crossing of the money demands, m^u (red curve) and m^b (blue curve), is illustrated in the right panel of Figure 2. The nonmonotonicity of $m^u - m^b$ with respect to s creates a nonmonotone relationship between bank entry and spreads. For low spreads, an increase in s reduces bank entry, whereas for high spreads it raises bank entry.

In summary, our model delivers a transmission mechanism from the policy rate to investment through two channels. There is an internal finance channel whereby an increase in s_t reduces entrepreneurs' holdings of liquid assets, which in turn reduces the share of total investment financed internally. This effect is asymmetric for banked and unbanked entrepreneurs and depends on the elasticity of $\psi(\textsc{l})$ and the difference between λ^b and λ^u. There is also an external finance channel, according to which an increase in s makes lending relationships more valuable when $m^u > m^b$, which raises bank profits when banks have market power and promotes loan creation.

4 Aftermath of a credit shock

We now study the dynamics of the economy following a credit supply shock under alternative monetary policies. We assume fundamentals are such that $m^u > m^b$ for all $s \in (0, \hat{s})$, which is the case, e.g., if $\lambda^u = \lambda^b$ or $\lambda^u < \lambda^b$, and the elasticity of $\psi'(\textsc{l})$ is sufficiently large. The economy starts at a steady state with $\ell_0 = \ell^s$ and $m^b > 0$. A banking crisis destroys a fraction of the lending relationships, $\ell_0^+ < \ell^s$. We illustrate the dynamics in a phase diagram for the continuous-time limit of our model (see Appendix A2 for derivations).[13]

4.1 Interest or liquidity targeting vs. forward guidance

We consider simple policies that illustrate the central trade-off of the policymaker between providing liquidity to unbanked entrepreneurs and promoting the creation of lending relationships.

[13] See Choi and Rocheteau (2020) for a detailed description of New Monetarist models in continuous time with methods and applications.

Interest rate targeting. The first policy consists of keeping the spread, s_t, constant over time so as to maintain an elastic supply of liquidity. The steady state is a saddle point with a unique saddle path, $\theta_t = \theta^s$ for all t, leading to it, as illustrated by the left panel of Figure 3. For any ℓ_0, the measure of relationships is given by

$$\ell_t = \ell^s + (\ell_0 - \ell^s)\, e^{-[\delta+\alpha(\theta^s)]t},$$

where $\ell^s = \alpha(\theta^s)/\,[\delta + \alpha(\theta^s)]$. The speed of recovery, $\delta + \alpha(\theta^s)$, increases with the interest rate spread, s. Aggregate liquidity is given by $\mathrm{M}_t = \ell_t m^b + (1 - \ell_t)m^u$, i.e.,

$$\mathrm{M}_t = \mathrm{M}^s + (\ell^s - \ell_0)\, e^{-[\delta+\alpha(\theta^s)]t}(m^u - m^b),$$

where $\mathrm{M}^s = \ell^s m^b + (1 - \ell^s)m^u$. Hence, aggregate liquidity jumps upward as the credit shock occurs and returns gradually to its steady-state value over time. Investment levels by banked and unbanked firms are unaffected by the shock, $k_t^u = k_0^u < k_t^b = k_0^b$, as illustrated in the bottom right panel of Figure 3. If $\lambda^u < \lambda^b$, then aggregate investment falls since the measure of unbanked firms is higher relative to the steady state, and those firms can only finance a fraction of the investment opportunities of banked firms. If $\lambda^u = \lambda^b$, then aggregate investment is unaffected because external financing crowds out internal financing one for one, by a similar logic as in Gu, Mattesini, and Wright (2016).

Aggregate liquidity targeting. Suppose next that aggregate liquidity, M_t, is held constant. Since m^u and m^b are decreasing in s and $m^u > m^b$, the market-clearing spread, $s(\ell, \mathrm{M})$, is decreasing in both ℓ and M for all ℓ and M$< k^*$ and $s(\ell,\mathrm{M}) = 0$ for all M$> k^*$. The θ-isocline is now downward sloping in (ℓ, θ)-space when M$< k^*$ and is horizontal if M$> k^*$. Intuitively, as ℓ increases, the aggregate demand for liquid assets decreases, and hence s decreases, which reduces the profitability of banks and bank entry. The steady state is a saddle point, and the saddle path is downward sloping if M$< k^*$, as illustrated by the top right panel of Figure 3.

If M$< k^*$, the interest rate spread and credit market tightness increase at the time of the credit crunch. As the economy recovers, both θ and s decrease and gradually return to their steady-state values. Investment by banked and unbanked firms drops initially, since s is higher, but recovers afterward. As a result, keeping M constant speeds up the formation of lending relationships (see bottom left panel of Figure 3) but does not accommodate the higher demand for liquidity created by the larger fraction of unbanked entrepreneurs, thereby reducing individual investment.

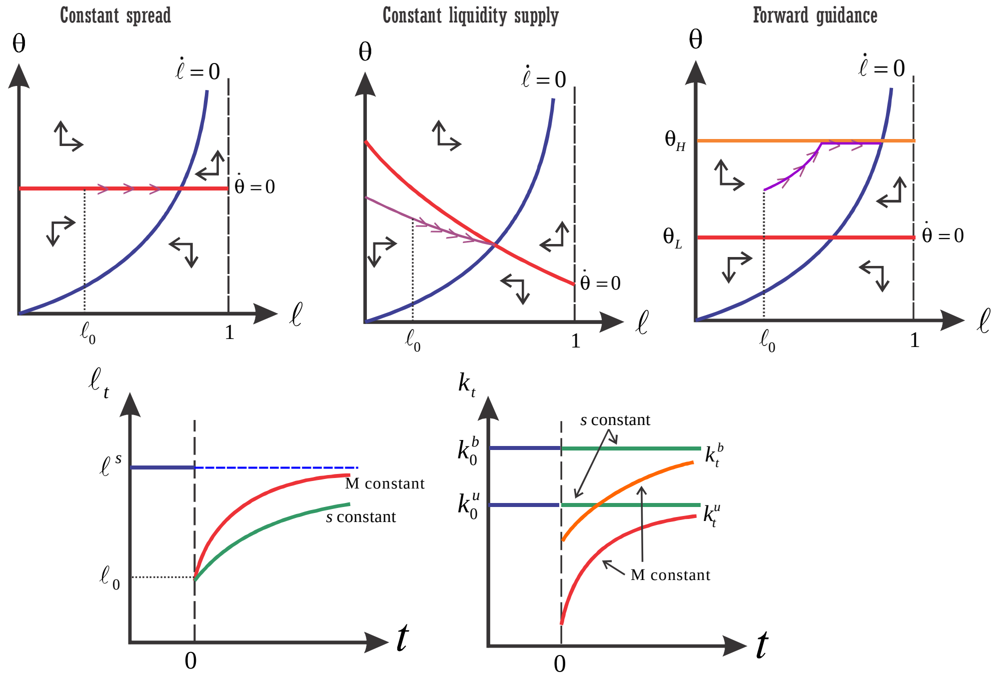

Figure 3: Transitional dynamics: phase diagram with constant s (top left), constant M (top middle), and forward guidance (top right); dynamics of lending relationships (bottom left) and investment (bottom right)

Forward guidance. The policymaker faces a trade-off between providing liquidity to unbanked entrepreneurs by keeping spreads low and giving incentives to banks to re-enter and rebuild relationships by raising spreads. In order to address both objectives, the policymaker can take advantage of banks' dynamic incentives by setting a low interest spread initially, $s(t) = s_L$ for all $t < T$, to allow unbanked entrepreneurs to self-insure at low cost, and by committing to raise the interest spread at some future date T, i.e., $s(t) = s_H > s_L$ for all $t > T$. We illustrate this policy in the top right panel of Figure 3. The low-spread regime corresponds to a low θ-isocline ($\theta = \theta_L$), and the high-spread regime corresponds to a high θ-isocline ($\theta = \theta_H$). The arrows of motion characterize the dynamic system when $t < T$. The equilibrium path can be obtained by moving backward in time.[14] For all $t > T$, the economy is on the horizontal saddle path corresponding to $s = s_H$, i.e., $\theta_t = \theta_H$. The path for the economy is continuous at $t = T$, i.e., $\theta_{T^-} = \theta_H$, and it reaches the θ_H-saddle path below. Finally, the trajectory of the economy starts at ℓ_0 with $\theta_0 > \theta_L$. Over the time interval $(0, T)$, market tightness rises until it reaches θ_H at time T. Aggregate liquidity

[14] Choi and Rocheteau (2020) describe the methodology to solve for equilibria of a continuous-time New Monetarist model with policy announcements.

increases initially due to high demand and low spreads, decreases gradually as ℓ_t increases, and jumps downward when the spread is raised.

4.2 Calibration

We calibrate the model to match moments on U.S. small businesses and their banking relationships from the 2003 National Survey of Small Business Finances (SSBF). We take the period length as a month and set $\rho = (1.04)^{\frac{1}{12}} - 1 = 0.0033$. We adopt the following functional forms: $\alpha(\theta) = \overline{\alpha}\theta/(1+\theta)$, where $\overline{\alpha} \in [0,1]$, $y(k) = k^a/a$ with $a = 0.3$, and $\psi(\mathrm{L}) = (B\mathrm{L})^{1+\xi}/(1+\xi)$, where $\xi > 1$. The parameters to calibrate are then $(\bar{\alpha}, \delta, \xi, B, a, \lambda^u, \lambda^b, s, \eta, \zeta)$.

Parameter	Value	Moment	Data	Model
Parameters Set Directly				
Discount rate (annual %), ρ	4.00			
Destruction rate, δ	0.012	Avg. length of credit rel. (years)	7.20	7.20
Production curvature, a	0.30	Capital share	0.30	0.30
Parameters Set Jointly		**Moments used in optimization**		
Matching efficiency, $\bar{\alpha}$	0.465	Share of banked firms	0.65	0.65
Productivity shock, unbanked, λ^u	0.029	Average 3 yr. rate of innovation	0.65	0.65
Productivity shock, banked, λ^b	0.034	Effect of relationship on innovation	1.17	1.17
External finance - curvature, ξ	15.68	Elasticity of m^u/m^b to s	11.98	11.98
External finance level, B	3.42	m^u/m^b	1.30	1.30
Bargaining power, η	0.084	Average NIM (%)	6.00	6.00
Bank entry cost, ζ	0.015	Optimal spread (% annual)	1.90	1.90

Table 1: Calibration summary: parameters and targets

We define a credit relationship as an open or revolving line of credit with a bank. In the 2003 SSBF, 65% of small businesses actively looking for or using external funding report being in a credit relationship with a bank, with an average duration of 86.44 months. We set $\bar{\alpha} = 0.47$ and $\delta = 1/86.44$ to generate $\ell = 0.65$ and an average length of relationship of 86.44 months. We interpret s as the user cost of holding an index of money-like assets, MSI-ALL, which includes currency, deposit accounts, and institutional and retail money market funds.[15] As our baseline, we target the optimal long-run spread under commitment to equal the average real user cost of MSI-ALL from 2002–2004 of 1.9%. This pins down the fixed cost of bank entry and gives $\zeta = 0.015$. We identify firms' money demand in the data by using the cash-to-assets ratio for small businesses

[15] The user cost is the spread between the rate of return of the MSI-ALL index and a benchmark rate equal to the maximum rate across short-term money market assets plus a liquidity premium of 100 basis points. See Anderson and Jones (2011) for details on constructing the MSI-ALL series and the rate of return and FRED series OCALLP.

from the 2003 SSBF. The user cost of cash is based on the Divisia monetary aggregate, MSI-ALL, developed by Barnett (1980).[16] We estimate money demand by banked and unbanked firms, controlling for various sources of firm heterogeneity, by running the following regression:

$$\log(m_{i,t}) = \beta_b D_{i,t} + e_u(1 - D_{i,t})s_t + e_b D_{i,t} s_t + X_{i,t} \cdot \gamma + \epsilon_{i,t}, \tag{25}$$

where $m_{i,t}$ is cash-assets of firm i in year t, s_t is the user cost of cash in year t (common across firms), $D_{i,t} \in \{0, 1\}$ is an indicator that equals one if firm i has access to a line of credit in year t, $X_{i,t}$ is a vector of controls, including a constant, that captures firm i's attributes and financial characteristics in year t, and $\epsilon_{i,t}$ is an error term.[17] In order to control for the selection bias according to which cash-rich firms do not need bank credit in the first place, we only focus on firms actively looking for or using bank credit. Firms in this sample completed the survey on different dates from 2003 to 2005. Hence, we match each SSBF sample with the user cost on the completed date. In the Supplementary Data Appendix, we document the definition of variables and report summary statistics.

We obtain $\exp(-\beta_b) = 1.29$, significant at the 1% level. Hence, small businesses in the SSBF who are not in a lending relationship hold approximately 30% more cash than banked firms do, controlling for monetary policy and various firm characteristics. The user cost semi-elasticities for the demand for liquid assets by unbanked and banked firms are $\partial \log(m^u)/\partial s = -26.68$ and $\partial \log(m^b)/\partial s = -38.66$. We report the remaining regression coefficients in the Supplementary Data Appendix.

We set (B, ξ) to target the percentage difference in cash holdings of unbanked to banked firms of 30%, $m^u/m^b = 1.3$, and the semi-elasticity of liquidity demanded by unbanked relative to banked firms with respect to the user cost of $\partial \log(m^u)/\partial s - \partial \log(m^b)/\partial s = 11.98$. This implies $\xi = 15.7$ and $B = 3.4$. In steady state, the cost of internal finance is $\psi(k^b - m^b) = 0.0006$, or 0.3% of the amount of credit provided. Further, to reach the high-spread regime requires an annual spread above $\hat{s} = 70\%$, which never occurs in any of the numerical experiments considered.

We set λ^u and λ^b using estimates in the literature on the frequency of firm innovation and the

[16] Cash is defined as "any immediately negotiable medium of exchange," which includes certificates of deposit (CDs), checks, demand deposits, money orders, and bank drafts. The user cost is the spread between the own rate of return from holding the portfolio of MSI-ALL and a benchmark rate that equals 100 basis points plus the maximum of the interest rate on short-term money market rates and the largest interest rate out of the components of MSI-ALL.

[17] Controls in the SSBF regressions include firm industry, urban or rural, firm size, firm type, as well as productivity related variables like ROA, growth, management, ownership, sales-assets ratio, owner's years of experience, and owner's level of education.

role of lending relationships. Studies have found that lending relationships generally increase the rate of innovation for both products and operational processes, however, the estimates vary across studies, geographies, and other firm characteristics. These studies find that, on average, 65% of firms report conducting a product or process innovation across three years and that a lending relationship increases the rate of innovation between 5% and 28%. We target the midpoint of this range, $\lambda^b/\lambda^u = 1.17$, and set λ^u to hit the average rate of innovation. This implies $\lambda^u = 0.029$ and $\lambda^b = 0.034$, or that unbanked and banked firms receive investment opportunities on average every 2.8 and 2.4 years, respectively.[18]

Finally, we set banks' bargaining power, η, to target the average annual NIM on small business loans. We use bank-level data from the FFIEC's Call Reports to measure NIM on small business loans. We define small business loans as loans less than $1 million and focus on banks that make more than half of their commercial and industrial (C&I) loans in amounts less than this cutoff.[19] The NIM for small business loans is measured as

$$NIM_{sb} = \underbrace{\frac{\text{interest and fee income on C\&I loans}}{\text{total C\&I loans}}}_{\text{loan rate}} - \underbrace{\frac{\text{total interest expense}}{\text{total assets}}}_{\text{funding cost}}.$$

The first term measures banks' return from business lending, while the second term represents banks' cost of funds. Since we do not directly observe the loan rate of small businesses in Call Reports, we proxy the small business loan rate by the first term. Note that most of the C&I loans of these banks are small business loans. We find a small business NIM of 6%. Matching this estimate gives $\eta = 0.08$.

In Appendix A7, we illustrate robustness to the key parameters of the model and moments targeted in the calibration.

Economy's response to a destruction of relationships. We consider different magnitudes for the size of the shock: a contraction in ℓ of 10%, 35%, and 60%. In our calibrated economy, ℓ falls from a steady state of 0.65 to 0.59, 0.42, and 0.26, respectively. These shock sizes correspond to different interpretations for the contraction in small business lending in the U.S. during the 2008

[18] Appendix A6 explains in detail how we map the empirical estimates in the literature to our model; Appendix A7 shows our results hold if we use the endpoints 5% and 28%.

[19] This definition of small business loans is widely used in the literature and regulations like the Community Reinvestment Act. The finding is similar if we focus on banks making more than 99% of their C&I loans less than $1 million.

banking crisis and recession.[20]

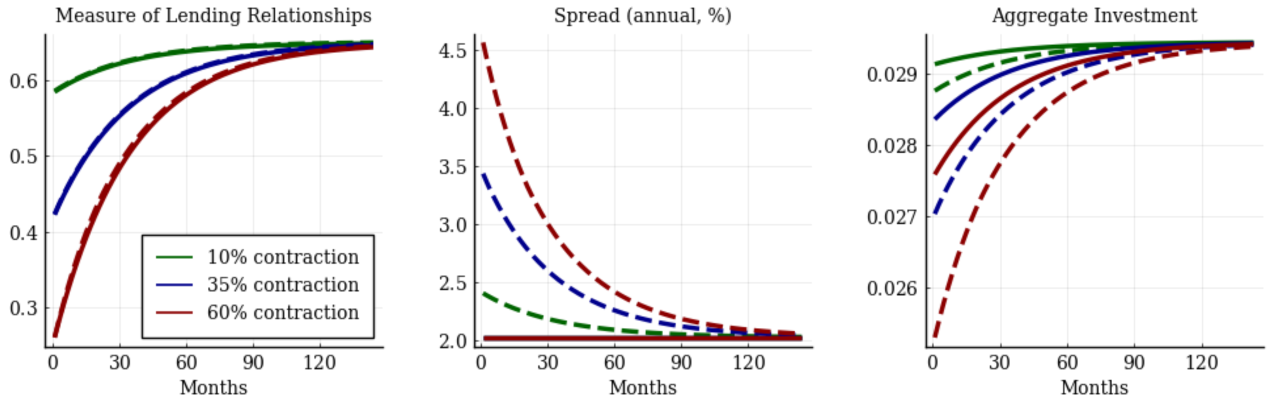

Figure 4: Dynamic response to a destruction of lending relationships: constant spread (solid) vs. constant liquidity (dashed)

Figure 4 shows the dynamic response of the calibrated model under two fixed monetary policy rules: the spread s remains constant over time (solid lines), and the aggregate supply of liquidity M remains constant (dashed lines). Those dynamics are qualitatively similar to those in Figure 3. Quantitatively, the policy that consists in keeping M constant generates a large decline in aggregate investment (by about 15% under the large shock), which is more than twice as large as the one obtained under a constant spread. The recovery in terms of lending relationships is slightly faster under a constant M than under a constant s.

5 Optimal monetary policy

We now analyze the optimal monetary policy following a credit crisis that destroys a fraction of lending relationships. In Appendix A1, we show any constrained-efficient allocation has $k_t^u = k_t^b = k^*$, $\mathrm{L}_t = 0$. It coincides with a decentralized equilibrium if the policymaker implements the Friedman rule, $s_t = 0$, to achieve optimal investment levels, and the Hosios condition holds, $\epsilon(\theta_t) = \eta$, to guarantee an efficient creation of lending relationships.[21] In Appendix A1, we also establish sufficient conditions under which the Friedman rule, $s_t = 0$ for all t, is suboptimal, i.e., if

[20] The contraction in lending relationships of 10% is in line with evidence from McCord and Prescott (2014) of a 14% decline in the number of commercial banks from 2007 to 2013. The larger contractions of 35% and 60% correspond roughly to the fall in the measure of small business loan originations of 40% reported in Chen, Hanson, and Stein (2017) and the fall in the total number of U.S. corporate loans of 60% as reported in Ivashina and Scharfstein (2010).

[21] See Appendix A1 for a formal proof. For a related result, see Berentsen, Rocheteau, and Shi (2007).

the difference between $\epsilon(\theta)$ and η is sufficiently large relative to $1/\xi$.[22]

5.1 The Ramsey problem

Suppose the policymaker chooses an infinite sequence of interest spreads to implement a decentralized equilibrium that maximizes social welfare. The policy path, $\{s_t\}_{t=1}^{\infty}$, is announced before the market for relationships opens in stage 2, and the policymaker commits to it. We write the Ramsey problem recursively by treating credit market tightness in every period $t \geq 1$, θ_t, as a state variable. The initial tightness, θ_0, is chosen to place the economy on the optimal path. Market tightness, θ_t, is interpreted as a promise to banks that determines their future profits. It must be honored in period $t+1$ by choosing s_{t+1} and θ_{t+1} consistent with the free entry condition, (21). The recursive planner's problem is

$$\begin{aligned} \widetilde{\mathbb{W}}(\ell_t, \theta_t) \quad = \quad & \max_{\theta_{t+1} \in \Gamma(\theta_t)} \left\{ -\zeta \theta_t (1-\ell_t) + \beta (1-\ell_{t+1}) \lambda^u \left[y(m^u_{t+1}) - m^u_{t+1} \right] \right. \\ & + \beta \ell_{t+1} \lambda^b \left[y\left(k^b_{t+1}\right) - k^b_{t+1} - \psi(\mathrm{L}_{t+1}) \right] \\ & \left. + \beta \widetilde{\mathbb{W}}(\ell_{t+1}, \theta_{t+1}) \right\}, \end{aligned} \tag{26}$$

where $\Gamma(\theta_t)$ is the set of values for θ_{t+1} consistent with (21) for some $s_{t+1} \in [0, +\infty)$. Given s_{t+1}, the quantities m^u_{t+1}, k^b_{t+1}, and L_{t+1} are obtained from (12) and (15).[23] The policymaker's problem at the beginning of time is

$$\mathbb{W}(\ell_0) = \max_{\theta_0 \in \Omega = [\underline{\theta}, \bar{\theta}]} \widetilde{\mathbb{W}}(\ell_0, \theta_0), \tag{27}$$

where $\underline{\theta}$ (respectively, $\bar{\theta}$) is the steady-state value of θ if $s = 0$ (respectively, $s = \infty$).

Figure 5 illustrates the optimal policy response following a banking shock that destroys lending relationships in the calibrated economy. The Ramsey solution lowers s_t at the onset of the crisis and then raises it above its long-run value (forward guidance) before gradually decreasing it to the stationary level. The hump-shaped path of the optimal spread (top left panel of Figure 5) causes a similar hump-shaped response in credit market tightness (top middle panel). While low spreads benefit unbanked firms by making internal finance less costly, they dampen bank profits and reduce the incentive to create new relationships. The use of forward guidance mitigates the effect of the current spread on relationship creation because banks' lifetime profits depend on the whole path of

[22] We do not allow the policymaker to make direct transfers to banks that participate in the credit market in order to correct for inefficiently low entry. Such transfers may not be feasible if the policymaker cannot distinguish between active and inactive banks in the credit market (i.e., one could create a bank but not search actively in the credit market).

[23] The existence and uniqueness of $\widetilde{\mathbb{W}}(\ell, \theta)$ is established in Appendix A1.

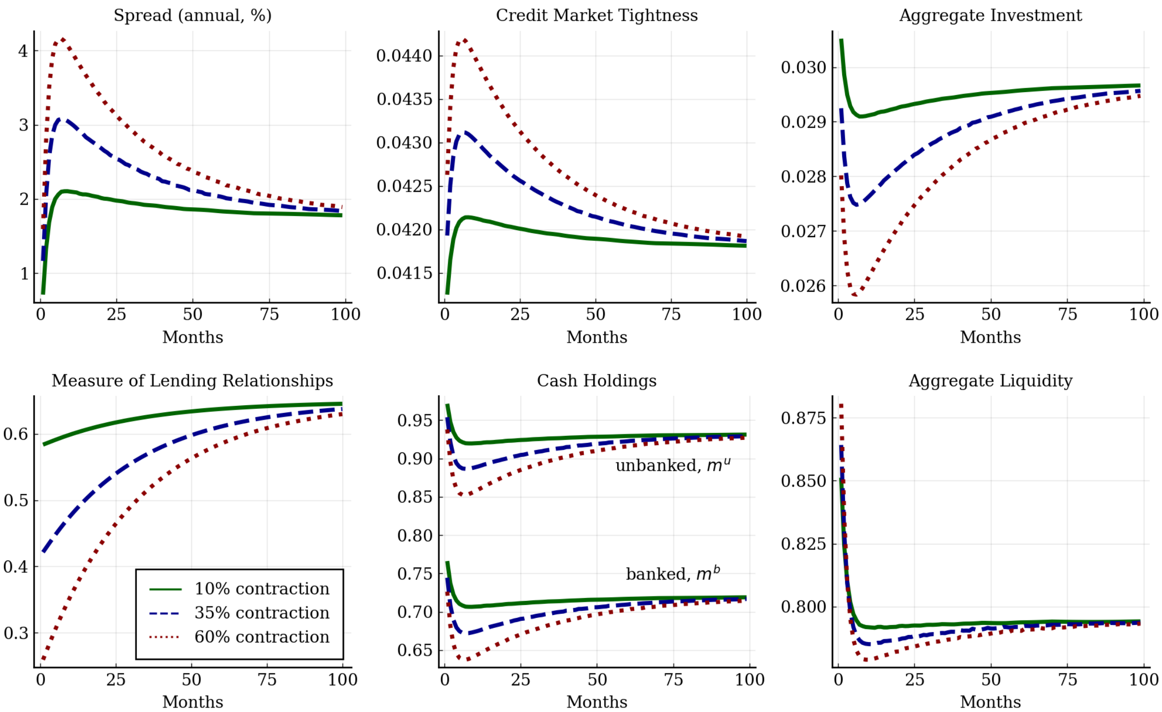

Figure 5: Optimal policy response with commitment to a destruction of lending relationships

future spreads. For smaller shocks, initial spreads are lower for longer and only slightly rise above their long-run level. However, for large shocks, spreads increase more rapidly and feature a more pronounced hump-shaped response.

Low spreads induce all firms to raise their holdings of liquid assets closer to the full insurance level $k^* = 1$ (bottom middle panel). Over time, the policymaker unwinds the initial expansion of M, leading to a sharper decline in banked firms' holdings relative to unbanked firms. Initially, the increase in liquidity counteracts the destruction of relationships causing aggregate investment to either rise in the case of small shocks or fall in the case of large shocks (top right panel). Afterwards, aggregate investment falls and reaches its lowest value after about 5 months. It recovers gradually as it increases towards its long-run value.

The Ramsey solution studied in this section is not bound by any past promises at time $t = 0$, i.e., it is free to select any equilibrium, in our context by choosing any θ_0. Woodford (1999, 2003) proposed a *timeless approach* amending the Ramsey solution to discipline the initial choice of equilibrium by taking into account past commitments. Since we start with an economy at the steady state, we impose that $\theta_0 = \theta^*$ where θ^* is credit market tightness at the steady state. We show in Appendix A5 that the results are qualitatively similar, i.e., the Ramsey solution under a timeless approach features a hump-shaped path for the spread. The constraint on the initial credit market tightness does impact the initial spread. For small shocks, s_0 is set just below its long-run value whereas for large shocks, s_0 is close to zero. Also, a noticeable difference is that aggregate investment under the timeless approach always falls following a destruction of relationships.

5.2 Optimal policy without commitment

We now relax the assumption of commitment altogether and assume the policymaker sets s_{t+1} in period t but cannot commit to $\{s_{t'}\}_{t'>t+1}$. As in Klein, Krusell, and Ríos-Rull (2008), the policymaker moves first by choosing s_{t+1}, and the private sector moves next by choosing θ_{t+1}, m^u_{t+1}, and m^b_{t+1}.

We restrict our attention to Markov-perfect equilibria. The policymaker's strategy consists of a spread s_{t+1} at the beginning of stage 2 of period t as a function of the economy's state, ℓ_t. From (12), $m^u_t = y'^{-1}\left[1 + s_t/\lambda^u\right]$, so the strategy of the policymaker can be represented by $m^u_{t+1} = \mathcal{K}\left(\ell_t\right)$. The strategy of banks to enter is expressed as $\theta_t = \Theta(\ell_t, m^u_{t+1})$, where Θ is implicitly defined by (21), i.e.,

$$\frac{\Theta(\ell_t, m^u_{t+1})}{\alpha\left[\Theta(\ell_t, m^u_{t+1})\right]} = \frac{\beta\eta\Delta\pi(m^u_{t+1})}{\zeta} - \beta(1-\eta)\Theta(\ell_{t+1}, m^u_{t+2}) + \beta(1-\delta)\frac{\Theta(\ell_{t+1}, m^u_{t+2})}{\alpha\left[\Theta(\ell_{t+1}, m^u_{t+2})\right]}, \tag{28}$$

where $\ell_{t+1} = (1-\delta)\ell_t + \alpha\left[\Theta(\ell_t, m^u_{t+1})\right](1-\ell_t)$ and $m^u_{t+2} = \mathcal{K}\left(\ell_{t+1}\right)$. When forming expectations about θ_{t+1}, banks anticipate the policymaker in period $t+1$ will adhere to his policy rule, $m^u_{t+2} = \mathcal{K}\left(\ell_{t+1}\right)$, and hence $\theta_{t+1} = \Theta\left[\ell_{t+1}, \mathcal{K}\left(\ell_{t+1}\right)\right]$. In equilibrium, $\theta_t = \Theta(\ell_t, m^u_{t+1})$ and $m^u_{t+1} = \mathcal{K}\left(\ell_t\right)$ are best responses to each other.

Given Θ, we determine $\mathcal{K}\left(\ell_t\right)$ recursively from

$$\begin{aligned}\mathcal{W}(\ell_t) \quad = \quad & \max_{m^u_{t+1}\in[0,k^*]} \left\{-\zeta(1-\ell_t)\Theta\left(\ell_t, m^u_{t+1}\right) + \beta(1-\ell_{t+1})\lambda^u\left[y(m^u_{t+1}) - m^u_{t+1}\right]\right. && (29)\\ & +\beta\ell_{t+1}\lambda^b\left[y\left(k^b_{t+1}\right) - k^b_{t+1} - \psi(\mathrm{L}_{t+1})\right] \\ & \left. +\beta\mathcal{W}\left(\ell_{t+1}\right)\right\}, && (30)\end{aligned}$$

where k^b_{t+1} and L_{t+1} can be expressed as functions of m^u_{t+1}. The entire transitional dynamics are computed numerically by devising a two-dimensional iteration described in Appendix A4. Figure 6 illustrates the optimal policy response under the benchmark calibration.

The policymaker can no longer commit to raise future s_t to counteract the effects of current s_t on banks' lifetime expected profits, as in the Ramsey problem. In the long run, it is optimal to set

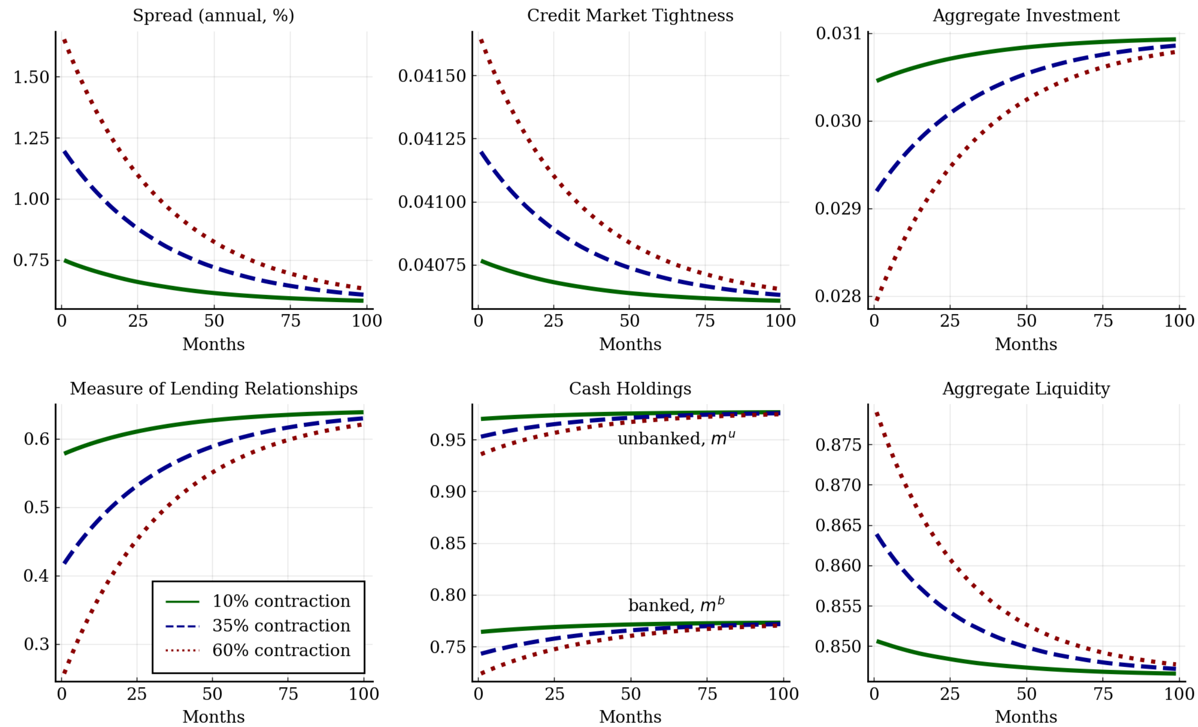

Figure 6: Optimal policy response without commitment to a destruction of lending relationships

spreads just above their lower bound of zero, which illustrates the bias of the policymaker toward low spreads. After a shock to banking relationships, the policymaker raises spreads at the outset of the credit crunch in order to rebuild relationships more quickly (top left panel of Figure 6). The initial increase in spreads is commensurate with the size of the shock, increasing more for larger shocks. As lending relationships recover, s_t is reduced gradually over time. Quantitatively, interest spreads early on are larger than the ones from the Ramsey solution when the shock is large enough, but future spreads are lower.

There are two effects on aggregate liquidity at the onset of the crisis. The increase in spreads lowers holdings of liquid assets for all firms, thereby decreasing M_t. However, the fall in ℓ from the credit shock increases M_t since $m^u > m^b$. Quantitatively, the second effect tends to dominate (bottom right panel), more so the larger the shock. In contrast to the Ramsey solution, aggregate investment falls at the time of the shock and recovers gradually (top right panel).

In our calibrated example, the half life of the recovery following a 60% contraction is 23 months under the Ramsey solution but 26 months under the no commitment policy. So, the inability to commit slows down the recovery by 3 months. The welfare loss from the lack of commitment ranges from 0.25% to 0.35% of foregone output, depending on the size of the shock.[24] These estimates are in the same ball park as those in Klein, Krussell, and Ríos-Rull (2008) who study time consistent capital income taxation and find that without commitment, steady-state consumption is about 0.5% lower relative to the commitment case.

6 Search for investment opportunities

We now endogenize λ^u and λ^b by assuming that both the entrepreneur and the bank exert some effort in stage 1, e^f and e^b, respectively, to search for and screen profit opportunities. The disutility of effort for both agents is $-e$. The probability of an investment opportunity, $\Lambda(E)$, where the joint effective effort, denoted E, is given by

$$E(e^f, e^b) = \left[(e^f)^\varepsilon + \kappa(e^b)^\varepsilon\right]^{\frac{1}{\varepsilon}}, \tag{31}$$

with $\varepsilon \in (0,1]$ and $\kappa \in (0,1)$. If we think of e as the effort to screen a flow of potential investment opportunities, then $\Lambda(E)$ is the flow of projects that are identified as profitable.

The problem of an unbanked entrepreneur is generalized as follows:

$$\pi^u(s_t) \equiv \max_{m_t \geq 0, e_t^f \geq 0} \left\{-s_t m_t - e_t^f + \Lambda(e_t^f) \max_{k_t \leq m_t} [y(k_t) - k_t]\right\}. \tag{32}$$

In addition to the cost of holding liquid assets, entrepreneurs now incur the cost of searching and screening investment opportunities. From (31), $e^b = 0$ implies $E^u = e^f$, and the optimal search effort of unbanked entrepreneurs solves

$$1 = \Lambda'(E_t^u)\left[y(k_t^u) - k_t^u\right]. \tag{33}$$

The investment probability is $\lambda^u = \Lambda(E^u)$, which decreases with s_t. By a similar reasoning (see Appendix A3 for details), the joint search effort in a relationship E^b is the solution to

$$\left(1 + \kappa^{\frac{1}{1-\varepsilon}}\right)^{\frac{\varepsilon-1}{\varepsilon}} = \Lambda'\left(E_t^b\right)\left[y(k_t^b) - k_t^b - \psi(k_t^b - m_t^b)\right], \tag{34}$$

where the individual efforts are given by $e^b = \kappa^{\frac{1}{1-\varepsilon}} e^f$ and $e^f = \left(1 + \kappa^{\frac{1}{1-\varepsilon}}\right)^{\frac{-1}{\varepsilon}} E^b$. The investment probability is $\lambda^b = \Lambda(E^b)$, which is also a function of s_t. The following proposition provides microfoundations for the ranking of λ^b and λ^u at the Friedman rule.

[24] We measure the welfare cost of not implementing the Ramsey solution as $1 - \mathcal{W}^p/\mathcal{W}^R$, where $\mathcal{W}^R$ is lifetime discounted output under the Ramsey policy, and $\mathcal{W}^p$ is lifetime discounted output under the optimal policy with no commitment. In Appendix A7, we report robustness checks on various parameters (e.g., η) and summarize how this welfare cost is affected.

Proposition 4 ***(Search for investment opportunities at the Friedman rule).*** *Suppose $s = 0$. If $\varepsilon = 1$, i.e., efforts are perfect substitutes, then $\lambda^b = \lambda^u$. If $\varepsilon \in (0, 1)$, i.e., efforts are imperfect substitutes, then $\lambda^b > \lambda^u$.*

As long as the search efforts of entrepreneurs and banks are not perfect substitutes, the investment probability of a banked entrepreneur is larger than the one of an unbanked entrepreneur at the Friedman rule, $\lambda^b > \lambda^u$. So, it suffices to avoid the knife-edge case $\varepsilon = 1$ to generate a wedge between λ^b and λ^u.

We recalibrate our model to check the implications of endogenizing λ^b and λ^u for optimal monetary policy. We adopt the following functional form, $\Lambda(e) = \min\{\gamma\sqrt{e}, 1\}$. Since this give us three free parameters but only two targets, $\lambda^u = \Lambda(E^u)$ and $\lambda^b = \Lambda(E^b)$, we simply set $\varepsilon = 0.5$ and replace (λ^u, λ^b) with (γ, κ) in our baseline calibration strategy. This implies the steady-state probabilities of investment opportunities for banked and unbanked firms match those from the baseline calibration, or $\Lambda(E^u) = 0.029$ and $\Lambda(E^b) = 0.034$. The remaining parameters are reported in Appendix A3.

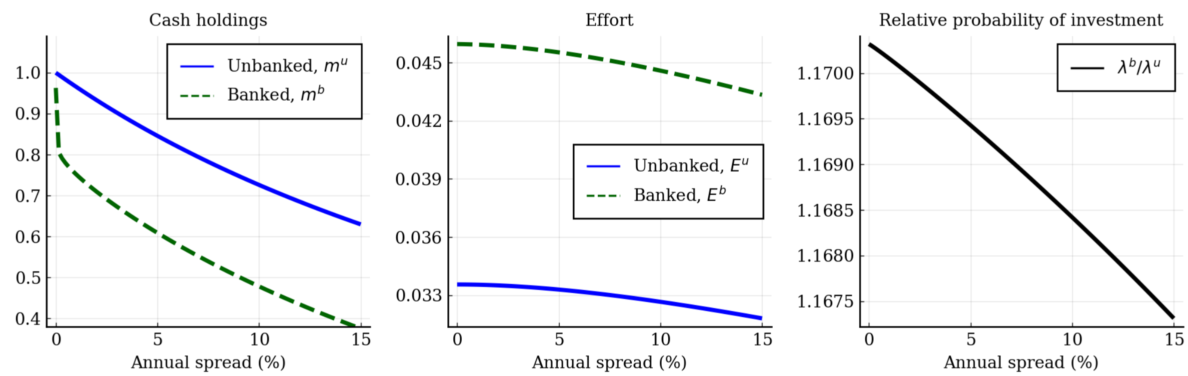

Figure 7: Response of cash holdings (left), effort (middle), and λ^b/λ^u (right) to s

The left and middle panels of Figure 7 illustrate the responses of cash holdings and search and screening efforts to changes in s_t. Since the two choices are complements, both fall as s_t increases. However, since the profit functions are flat close to the Friedman rule ($s_t = 0$), search ad screening efforts are relatively inelastic to changes in s_t when s_t is close to zero. The right panel shows the wedge, λ^b/λ^u, declines slightly as s_t increases away from the Friedman rule.

We show in Appendix A3 the optimal policy response to a destruction of lending relationships is qualitatively similar to the one in the model with exogenous λ. Quantitatively, under commitment, endogenizing investment opportunities leads to a more pronounced hump-shaped response in spreads, peaking 0.9 percentage points higher, relative to our benchmark case with exogenous λ^u and λ^b. Without commitment, the policymaker initially increases s_t by 0.1 percentage points more compared to when λ is fixed. In both cases, aggregate investment falls by more and takes longer to recover when λ^b and λ^u are endogenous, though the differences are small.

7 Conclusion

We argue in this paper that the formation of lending relationships is critical for small businesses to finance their investment opportunities. As the formation of these relationships can be influenced by monetary policy, we developed a general equilibrium model of corporate finance that formalizes this transmission mechanism, building on recent theories of money demand under idiosyncratic risk and financial intermediation in over-the-counter markets. We use our model to study the optimal response of the monetary authority following a banking crisis described as an exogenous destruction of a fraction of the existing lending relationships. We consider different assumptions regarding the policymaker's power to commit to setting a time path of interest spreads.

If the policymaker can commit over an infinite time horizon, the optimal policy involves "forward guidance": the interest spread is set close to its lower bound at the outset of the crisis and increases over time as the economy recovers. It is this promise of high future spreads that provides banks incentives to keep creating lending relationships, even in a low spread environment. However, such promises are not time consistent. If the policymaker cannot commit more than one period ahead, then the interest rate spread is persistently low, and the recession is more prolonged.

Our model of lending relationships and corporate finance can be extended in several ways. For instance, one could relax the assumption that banks can fully enforce repayment to study imperfect pledgeability of firms' returns and its relation with monetary policy (e.g., as in Rocheteau, Wright, and Zhang, 2018). One could introduce banks' limited commitment and analyze the dynamic contracting problem in the credit market (e.g., as in Bethune, Hu, and Rocheteau, 2017) or agency problems between firms and banks to capture additional benefits of lending relationships (e.g., Hachem, 2011; Boualam, 2017). It would be fruitful to develop a life-cycle version of our model to explain firms' cash accumulation patterns and their interaction with long-term credit lines. Our model of relationship lending could also be applied in other institutional contexts, like the interbank market (e.g., Brauning and Fecht, 2016). Last, but not least, while we focused on the lending channel of monetary policy when banks have market power, it would be useful to add an imperfectly competitive market for bank deposits, as suggested by Drechsler, Savov, and Schnabl (2017), in order to obtain a more complete description of the transmission mechanism of monetary policy (e.g., Choi and Rocheteau, 2021).

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Appendix A1: Proofs of Propositions and Lemmas

Proof of Proposition 1. We first compute the surplus of being in a lending relationship for entrepreneurs and banks. Recall from (1) and (2) that the lifetime expected utilities of banked and unbanked entrepreneurs with wealth ω in the last stage of period t solve:

$$W_t^e(\omega) = \omega - \frac{m_{t+1}^u}{1+r_{t+1}} + \beta U_{t+1}^e(m_{t+1}^u), \tag{35}$$

$$X_t^e(\omega) = \omega - \frac{m_{t+1}^b}{1+r_{t+1}} + \beta Z_{t+1}^e(m_{t+1}^b). \tag{36}$$

The surplus of a banked entrepreneur is defined as $\mathcal{S}_t^e = \left[X_{t-1}^e(0) - W_{t-1}^e(0)\right]/\beta$. Substituting Z_t^e by its expression given by (8), i.e.,

$$Z_t^e(m_t^b) = m_t^b - \phi_t + \lambda^b\left[y(k_t^b) - k_t^b\right] + \delta W_t^e(0) + (1-\delta)X_t^e(0),$$

into (36) and subtracting $W_{t-1}^e(0)$, we obtain:

$$\begin{aligned}\beta\mathcal{S}_t^e = {} & \beta\left\{-s_t m_t^b - \phi_t + \lambda^b\left[y(k_t^b) - k_t^b\right]\right\} \\ & +(1-\delta)\beta\left[X_t^e(0) - W_t^e(0)\right] + \beta W_t^e(0) - W_{t-1}^e(0).\end{aligned} \tag{37}$$

From (6), for all $m_t^u \leq k^*$,

$$U_t^e(m_t^u) = \lambda^u\left[y(m_t^u) - m_t^u\right] + m_t^u + V_t^e(0),$$

which we substitute into (35) to express $W_{t-1}^e(0)$ as:

$$W_{t-1}^e(0) = \beta\{-s_t m_t^u + \lambda^u\left[y(m_t^u) - m_t^u\right]\} + \beta V_t^e(0).$$

Substituting $W_{t-1}^e(0)$ into (37) and dividing both sides by β we obtain:

$$\begin{aligned}\mathcal{S}_t^e = {} & -\phi_t - s_t m_t^b + \lambda^b\left[y(k_t^b) - k_t^b\right] \\ & - \left[\pi_t^u + V_t^e(0) - W_t^e(0)\right] + (1-\delta)\beta\mathcal{S}_{t+1}^e.\end{aligned} \tag{38}$$

From (9), the surplus of the bank solves

$$\mathcal{S}_t^b = \phi_t - \lambda^b\psi\left(k_t^b - m_t^b\right) + \beta(1-\delta)S_{t+1}^b, \tag{39}$$

where m_t^b is the entrepreneur's down payment on the loan and hence $\mathrm{L}_t = k_t^b - m_t^b$ is the loan size. Summing (38) and (39), the total surplus of a lending relationship, $\mathcal{S}_t = \mathcal{S}_t^e + \mathcal{S}_t^b$, solves

$$\begin{aligned}\mathcal{S}_t = {} & -s_t m_t^b + \lambda^b\left[y(k_t^b) - k_t^b - \psi\left(k_t^b - m_t^b\right)\right] \\ & - \left[\pi_t^u + V_t^e(0) - W_t^e(0)\right] + (1-\delta)\beta\mathcal{S}_{t+1}.\end{aligned} \tag{40}$$

A lending relationship contract negotiated at time $t-1$ is

$$\{k_{t+\tau}^b, m_{t+\tau}^b, \phi_{t+\tau}\}_{\tau=0}^{\infty} \in \arg\max[\mathcal{S}_t^b]^{\eta}[\mathcal{S}_t^e]^{1-\eta}. \tag{41}$$

Given the linearity of $\mathcal{S}_t^e$ and $\mathcal{S}_t^b$ in ϕ_t, a solution is such that $\{k_{t+\tau}^b, m_{t+\tau}^b\}_{\tau=0}^{\infty} \in \arg\max \left\{\mathcal{S}_t^b + \mathcal{S}_t^e\right\}$ and $\phi_t \in \arg\max[\mathcal{S}_t^b]^{\eta}[\mathcal{S}_t^e]^{1-\eta}$. Maximizing $\mathcal{S}_t$ with respect to k_t^b and m_t^b gives:

$$\begin{aligned} y'(k_t^b) - 1 - \psi'\left(k_t^b - m_t^b\right) &= 0 \\ -s_t + \lambda^b \psi'\left(k_t^b - m_t^b\right) &\leq 0, \quad \text{“=” if } m_t^b > 0. \end{aligned}$$

The sequence of intermediation fees solves $\mathcal{S}_{t+\tau}^b = \eta \mathcal{S}_{t+\tau}$ for all τ, where $\mathcal{S}_t^b$ obeys (39). Using the definition of $\pi^b(s_t)$ in (17) we can reexpress $\mathcal{S}_t$ as:

$$\mathcal{S}_t = \pi^b(s_t) - \pi^u(s_t) - [V_t^e(0) - W_t^e(0)] + (1-\delta)\beta \mathcal{S}_{t+1}. \tag{42}$$

Solving for ϕ_t gives

$$\phi_t = \lambda^b \psi\left(k_t^b - m_t^b\right) + \eta\left[\pi^b(s_t) - \pi^u(s_t)\right] - \eta\left[V_t^e(0) - W_t^e(0)\right].$$

Using that from (3), $V_t^e(0) = \alpha_t X_t^e(0) + (1-\alpha_t) W_t^e(0)$, it follows that $V_t^e(0) - W_t^e(0) = \alpha_t\left[X_t^e(0) - W_t^e(0)\right] = \beta\alpha_t \mathcal{S}_{t+1}^e$. From the bargaining, $\mathcal{S}_{t+1}^e = (1-\eta)\mathcal{S}_{t+1}^b/\eta$ and from the free-entry condition in (4), $\mathcal{S}_{t+1}^b = \zeta\theta_t/(\beta\alpha_t)$. Putting all this together, we obtain (16). ■

Proof of Lemma 1. From (3), $V_t^e(0) - W_t^e(0) = \alpha_t \beta \mathcal{S}_{t+1}^e = \alpha_t(1-\eta)\beta\mathcal{S}_{t+1}$ where we used the generalized Nash solution, i.e. $\mathcal{S}_{t+1}^e = (1-\eta)\mathcal{S}_{t+1}$. Substituting into (42) to obtain:

$$\mathcal{S}_t = \Delta\pi(s_t) - \alpha_t(1-\eta)\beta\mathcal{S}_{t+1} + (1-\delta)\beta\mathcal{S}_{t+1}. \tag{43}$$

The first two terms on the right side of (43), $\Delta\pi(s_t) - \alpha_t(1-\eta)\beta\mathcal{S}_{t+1}$, represent the flow surplus from a lending relationship: it is the increase in the expected profits of the entrepreneur from having access to external finance net of the entrepreneur's outside option. From (4), assuming positive entry in equilibrium, $V_t^b = V_{t+1}^b = 0$ implies $\zeta\theta_t = \alpha_t\eta\beta\mathcal{S}_{t+1}$. Substituting $\mathcal{S}_{t+1} = \zeta\theta_t/\left(\eta\beta\alpha_t\right)$ into (43) gives (21). ■

Proof of Proposition 2. From (21) with $\theta_t = \theta_{t+1} = \theta$, steady-state credit market tightness is the unique solution to

$$(\rho+\delta)\frac{\theta}{\alpha(\theta)} + (1-\eta)\theta = \frac{\eta\left[\pi^b(s) - \pi^u(s)\right]}{\zeta}. \tag{44}$$

Using that the left side is increasing in θ and $\lim_{\theta\to 0}\theta/\alpha(\theta) = 1$, (44) admits a positive solution if

$$(\rho+\delta)\zeta < \eta\left[\pi^b(s) - \pi^u(s)\right].$$

By differentiating (44) and using that $\pi^{b\prime}(s) = -m^b$ and $\pi^{u\prime}(s) = -m^u$ we obtain:

$$\frac{\partial\theta}{\partial s} = \eta\frac{\left(m^u - m^b\right)}{\zeta}\left\{(\rho+\delta)\frac{[1-\epsilon(\theta)]}{\alpha(\theta)} + (1-\eta)\right\}^{-1}. \tag{45}$$

From (12) and (15), if $\lambda^b = \lambda^u = \lambda$, then $m^u > m^b$ for all $s > 0$. Hence, from the expression above, $\partial\theta/\partial s > 0$ for all $s > 0$.

Given θ, closed-form solutions for $(\ell, m^u, m^b, k^b, \phi)$ are obtained recursively as follows:

$$\ell = \frac{\alpha(\theta)}{\delta + \alpha(\theta)} \tag{46}$$

$$\frac{s}{\lambda} = y'(m^u) - 1 \tag{47}$$

$$m^b = \max\left\{m^u - \psi'^{-1}\left(\frac{s}{\lambda}\right), 0\right\} \tag{48}$$

$$k^b = \max\left\{m^u, \hat{k}\right\} \tag{49}$$

$$\phi = \lambda\psi(k^b - m^b) + \eta\left[\pi^b(s) - \pi^u(s)\right] - (1-\eta)\zeta\theta, \tag{50}$$

where $\hat{k}$ has been defined as the solution to $y'(\hat{k}) = 1 + \psi'\left(\hat{k}\right)$. Equation (46) is obtained from (22). Equation (47) corresponds to (12). Equation (48) corresponds to (15) where we used that if $m^b > 0$ then $k^b = k^u = m^u$ and hence $\psi'\left(k^b - m^b\right) = s/\lambda$. By taking the inverse of ψ', $m^b = m^u - \psi'^{-1}(s/\lambda)$. If $m^b = 0$, then $y'(k^b) - 1 = \psi'\left(k^b\right) \leq s/\lambda = y'(m^u) - 1$. It follows that $m^u \leq k^b \leq \psi'^{-1}(s/\lambda)$ where $k^b = \hat{k}$. This gives (49). Finally, (50) is obtained from (16).

The low-spread regime corresponds to the case where the constraint $m^b \geq 0$ does not bind. Hence $k^b = k^u = m^u$ and $m^b = m^u - \psi'^{-1}(s/\lambda)$, which corresponds to (24). The condition $m^b \geq 0$ for all s such that $m^u \geq \psi'^{-1}(s/\lambda)$, i.e., $y'^{-1}(1 + s/\lambda) \geq \psi'^{-1}(s/\lambda)$. The left side is decreasing in s while the right side is increasing in s, so there is a threshold $\hat{s}$ such that the inequality holds for all $s \leq \hat{s}$. The threshold solves $y'^{-1}(1 + \hat{s}/\lambda) = \psi'^{-1}(\hat{s}/\lambda)$, i.e., $y'(\hat{k}) - 1 = \psi'\left(\hat{k}\right) = \hat{s}/\lambda$. The high-spread regime corresponds to the case where the constraint $m^b \geq 0$ binds, in which case $k^b = \hat{k}$ and $s > \hat{s}$.

Finally, in the neighborhood of $s = 0$, from (47) and (48), $m^u = m^b \approx k^*$. From (45), $\partial\theta/\partial s \approx 0$. Differentiating (47) and (48),

$$\frac{\partial m^b}{\partial s} = \frac{1}{\lambda y''(k^*)} - \frac{1}{\lambda\psi''(0)} < \frac{\partial m^u}{\partial s} = \frac{1}{\lambda y''(k^*)} < 0.$$

■

Proof of Proposition 3. From (12) and (15), m^b and m^u are continuous functions of s such that $m^b = m^u = k^*$ when $s = 0$. If $\partial\left(m^b - m^u\right)/\partial s > 0$ when evaluated at $s = 0^+$ then there exists $s_0 > 0$ such that $m^b > m^u$ for all $s < s_0$. By differentiating the FOC (15),

$$\frac{\partial m^b}{\partial s} = \frac{1}{y''(k^b)\lambda^b} - \frac{1}{\psi''(k^b - m^b)\lambda^b}.$$

Using that $\psi''(k^b - m^b) = \psi_0(k^b - m^b)^{\xi-1}/\xi$, in the neighborhood of $s = 0^+$,

$$\frac{\partial m^b}{\partial s} = \begin{cases} \frac{1}{y''(k^*)\lambda^b} & \\ \frac{1}{y''(k^*)\lambda^b} - \frac{1}{\psi_0\lambda^b} & \text{if } \xi \quad \begin{matrix} < 1 \\ = 1 \\ > 1. \end{matrix} \\ -\infty & \end{cases}$$

The condition $\partial m^b/\partial s > \partial m^u/\partial s = 1/\left[y''(k^*)\lambda^u\right]$ holds if $\xi < 1$; it is equivalent to

$$\frac{\lambda^b - \lambda^u}{\lambda^u} > \frac{-y''(k^*)}{\psi_0}$$

if $\xi = 1$; it does not hold if $\xi > 1$. Hence, if $\xi < 1$ or $\xi = 1$ and $\left(\lambda^b - \lambda^u\right)/\lambda^u > -y''(k^*)/\psi_0$, then there exists $s_0 > 0$ such that $m^b > m^u$ for all $s < s_0$. By the same reasoning as in the proof of Proposition 2, for all $s > \hat{s} \equiv \lambda^b \psi'\left(\hat{k}\right)$, $m^b = 0 < m^u$. Hence, $s_1 \leq \hat{s}$. The effect of a change in s on market tightness is obtained from (45) in the proof of Proposition 2, according to which $\partial\theta/\partial s \sim \Delta\pi'(s) = \left(m^u - m^b\right)$. ■

Proposition 5 ***(Implementing constrained efficient allocations.)*** *The equilibrium achieves the constrained-efficient allocation if and only if $s_t = 0$ and $\epsilon(\theta_t) = \eta$ for all t.*

Proof of Proposition 5. We measure social welfare starting in stage 2 of period 0 as the discounted sum of aggregate output flows net of the costs associated with production, intermediation, and bank entry: $\mathbb{W}(\ell_0) = \sum_{t=0}^{\infty} \beta^t \mathcal{W}_t$ where the period welfare is

$$\begin{aligned} \mathcal{W}_t &= -\zeta\theta_t(1-\ell_t) + \beta(1-\ell_{t+1})\lambda^u\left[y\left(k^u_{t+1}\right) - k^u_{t+1}\right] \\ &\quad + \beta\ell_{t+1}\lambda^b\left[y(k^b_{t+1}) - k^b_{t+1} - \psi\left(\mathrm{L}_{t+1}\right)\right]. \end{aligned} \tag{51}$$

The first term on the right side of (51) represents banks' entry costs in the relationship lending market; the second term represents the profits of unbanked entrepreneurs; and the third term represents the profits of banked entrepreneurs net of the costs of external finance. The planner chooses $\left\{\theta_t, k^u_{t+1}, k^b_{t+1}, \mathrm{L}_{t+1}\right\}_{t=0}^{\infty}$ to maximize $\mathbb{W}(\ell_0)$ subject to the constraint imposed by the matching technology, $\alpha(\theta_t)$.

From the maximization of (51) with respect to $(k^u_{t+1}, k^b_{t+1}, \hat{k}^b_{t+1}, \mathrm{L}_{t+1})$, any constrained-efficient allocations satisfy:

$$k^u_{t+1} = k^b_{t+1} = k^* \tag{52}$$

$$\mathrm{L}_{t+1} = 0. \tag{53}$$

According to (52), the planner chooses the first-best level of investment, k^*. In that case, from (53), the loan size is zero. The comparison of the equilibrium conditions (12) and (15) with (52) shows that a necessary condition for the implementation of a constrained-efficient allocation is $s_{t+1} = 0$ for all t.

Using (52), we write the planner's problem recursively as

$$\begin{aligned} \mathbb{W}(\ell_t) &= \max_{\theta_t \geq 0}\left\{-\zeta\theta_t(1-\ell_t) + \beta(1-\ell_{t+1})\lambda^u\left[y\left(k^*\right) - k^*\right]\right. \\ &\quad \left. + \beta\ell_{t+1}\lambda^b\left[y(k^*) - k^*\right] + \beta\mathbb{W}(\ell_{t+1})\right\}, \end{aligned} \tag{54}$$

where $\ell_{t+1} = (1-\delta)\ell_t + \alpha(\theta_t)(1-\ell_t)$. Assuming an interior solution, the planner's optimality conditions are:

$$\zeta = \alpha'(\theta_t)\beta\omega_t \tag{55}$$

$$\omega_t = \left(\lambda^b - \lambda^u\right)\left[y(k^*) - k^*\right] + \beta\left\{1 - \delta - \alpha(\theta_{t+1})\left[1 - \epsilon(\theta_{t+1})\right]\right\}\omega_{t+1}, \tag{56}$$

where $\omega_t = (1+\rho)\left[\mathbb{W}'(\ell_t) - \zeta\theta_t\right]/\left[1 - \delta - \alpha(\theta_t)\right]$. From (21), in equilibrium the free-entry condition for banks when $s_{t+1} = 0$ is

$$\zeta = \frac{\alpha(\theta_t)}{\theta_t}\beta\eta\mathcal{S}_{t+1},$$

where

$$\mathcal{S}_{t+1} = \left(\lambda^b - \lambda^u\right)[y(k^*) - k^*] + \beta\left[(1-\delta) - \alpha(\theta_{t+1})(1-\eta)\right]\mathcal{S}_{t+2}.$$

This equilibrium condition coincides with the planner's optimality conditions, (55) and (56), if and only if $\epsilon(\theta_t) = \eta$. ■

Proposition 6 ***(Suboptimality of the Friedman rule.)*** *Suppose $\lambda^u < \lambda^b$, $\psi(\mathrm{L}) = \mathrm{L}^{1+\xi}/(1+\xi)$ with $\xi > 1$, and*

$$\zeta < \frac{\alpha'(0)\eta\left(\lambda^b - \lambda^u\right)[y(k^*) - k^*]}{\rho + \delta}. \tag{57}$$

It is optimal to deviate from $s_t \equiv 0$ if

$$\frac{\epsilon(\underline{\theta}) - \eta}{1 - \epsilon(\underline{\theta})} > \left[\frac{(1-\delta)\ell_0}{\alpha(\underline{\theta})(1-\ell_0)} + 1\right]\frac{1}{\xi}, \tag{58}$$

where $\underline{\theta}$ is steady-state credit market tightness at the Friedman rule.

Proof of Proposition 6. The economy starts with ℓ_0 lending relationships. We measure social welfare in the second stage of $t = 0$, before banks make entry decisions and entrepreneurs make portfolio decisions, by $\mathbb{W}_0 = \sum_{t=1}^{\infty} \beta^t \mathcal{W}_t$ where

$$\begin{aligned}\mathcal{W}_t \quad = \quad & -(1+\rho)\zeta(1-\ell_{t-1})\theta_{t-1} + (1-\ell_t)\lambda^u\left[y\left(k_t^u\right) - k_t^u\right] \\ & + \ell_t\lambda^b\left\{\left[y(k_t^b) - k_t^b - \psi\left(k_t^b - m_t^b\right)\right]\right\}.\end{aligned} \tag{59}$$

The first term on the RHS is the entry cost of banks in period $t-1$ where $(1-\ell_{t-1})\theta_{t-1}$ is the measure of banks entering. The following terms are the entrepreneurs' profits net of banks' monitoring costs in period t. (Relative to Proposition 5, $\mathcal{W}_t$ has been scaled up by $(1+\rho)$.)

We consider a small deviation of the interest rate spread from $s_1 = 0$. For $t \geq 2$, $s_t = 0$. As a result, for all $t \geq 1$, $\theta_t = \underline{\theta}$ solution to

$$(\rho + \delta)\frac{\underline{\theta}}{\alpha(\underline{\theta})} + (1-\eta)\underline{\theta} = \frac{\eta\left(\lambda^b - \lambda^u\right)[y(k^*) - k^*]}{\zeta} \tag{60}$$

where we used that $\pi^b(0) = \lambda^b[y(k^*) - k^*]$ and $\pi^u(0) = \lambda^u[y(k^*) - k^*]$. From (57), $\underline{\theta} > 0$. For all $t \geq 2$, $m_t^u = m_t^b = k^*$. From (22), the measure of lending relationships solves:

$$\begin{aligned}\ell_1 \quad &= \quad (1-\delta)\ell_0 + \alpha(\theta_0)(1-\ell_0) & (61)\\ \ell_t \quad &= \quad \underline{\ell} + (\ell_1 - \underline{\ell})[1 - \delta - \alpha(\underline{\theta})]^{t-1} \quad \text{for all } t \geq 1, & (62)\end{aligned}$$

where we have used that ℓ_t is the solution to the following linear, first-order difference equation, $\ell_{t+1} = (1-\delta)\ell_t + \alpha(\underline{\theta})(1-\ell_t)$, with initial condition ℓ_1. The long-run solution is $\underline{\ell} = \alpha(\underline{\theta})/[\delta + \alpha(\underline{\theta})]$. The welfare starting in the second stage of $t = 1$ is measured by $\mathbb{W}_1^0 = \sum_{t=2}^{\infty} \beta^{t-2}\mathcal{W}_t^0$ where

$$\mathcal{W}_t^0 = -(1+\rho)\zeta(1-\ell_{t-1})\underline{\theta} + \lambda^u[y(k^*) - k^*] + \ell_t(\lambda^b - \lambda^u)[y(k^*) - k^*],$$

where we have used that $\theta_{t-1} = \underline{\theta}$ and $m_t^u = m_t^b = k^*$. First, we rearrange the terms of the sum to rewrite $\mathbb{W}_1^0$ as follows:

$$\begin{aligned}\mathbb{W}_1^0 &= (1+\rho)\zeta\ell_1\underline{\theta} + \sum_{t=2}^{\infty}\beta^{t-2}\left\{-(1+\rho)\zeta\underline{\theta} + \lambda^u\left[y\left(k^*\right) - k^*\right]\right\} \\ &\quad + \sum_{t=2}^{\infty}\beta^{t-2}\ell_t\left\{\zeta\underline{\theta} + \left(\lambda^b - \lambda^u\right)\left[y(k^*) - k^*\right]\right\}. \end{aligned} \tag{63}$$

The second term on the right side of (63) is equal to:

$$\sum_{t=2}^{\infty}\beta^{t-2}\left\{-(1+\rho)\zeta\underline{\theta} + \lambda^u\left[y\left(k^*\right) - k^*\right]\right\} = \frac{-(1+\rho)\zeta\underline{\theta} + \lambda^u\left[y\left(k^*\right) - k^*\right]}{1-\beta}. \tag{64}$$

Using that

$$\begin{aligned}\sum_{t=2}^{\infty}\beta^{t-2}\ell_t &= \sum_{t=2}^{\infty}\beta^{t-2}\left\{\underline{\ell} + (\ell_1 - \underline{\ell})[1-\delta-\alpha(\underline{\theta})]^{t-1}\right\} \\ &= \frac{\underline{\ell}}{1-\beta} + \frac{(\ell_1 - \underline{\ell})[1-\delta-\alpha(\underline{\theta})]}{1-\beta[1-\delta-\alpha(\underline{\theta})]},\end{aligned}$$

the third term on the right side of (63) is equal to:

$$\begin{aligned}&\sum_{t=2}^{\infty}\beta^{t-2}\ell_t\left\{\zeta\underline{\theta} + \left(\lambda^b - \lambda^u\right)\left[y(k^*) - k^*\right]\right\} = \\ &\left\{\zeta\underline{\theta} + \left(\lambda^b - \lambda^u\right)\left[y(k^*) - k^*\right]\right\}\left(\frac{\underline{\ell}}{1-\beta} + \frac{(\ell_1 - \underline{\ell})[1-\delta-\alpha(\underline{\theta})]}{1-\beta[1-\delta-\alpha(\underline{\theta})]}\right).\end{aligned} \tag{65}$$

Substituting (64) and (65) into (63), and after some calculation:

$$\begin{aligned}\mathbb{W}_1^0 &= \frac{-(1-\underline{\ell})(1+\rho)\zeta\underline{\theta} + \lambda^u\left[y\left(k^*\right) - k^*\right] + \underline{\ell}\left(\lambda^b - \lambda^u\right)\left[y(k^*) - k^*\right]}{1-\beta} \\ &\quad + \frac{(1+\rho)\zeta\underline{\theta} + [1-\delta-\alpha(\underline{\theta})]\left(\lambda^b - \lambda^u\right)\left[y(k^*) - k^*\right]}{1-\beta[1-\delta-\alpha(\underline{\theta})]}(\ell_1 - \underline{\ell}).\end{aligned} \tag{66}$$

We are now in position to measure welfare from $t = 0$:

$$\begin{aligned}(1+\rho)\mathbb{W}_0 &= \mathcal{W}_1 + \beta\mathbb{W}_1^0 \\ &= -(1+\rho)\zeta(1-\ell_0)\theta_0 + (1-\ell_1)\lambda^u\left[y\left(k_1^u\right) - k_1^u\right] \\ &\quad + \ell_1\lambda^b\left[y(k_1^b) - k_1^b - \psi\left(k_1^b - m_1^b\right)\right] \\ &\quad + \frac{\zeta\underline{\theta} + \beta[1-\delta-\alpha(\underline{\theta})]\left(\lambda^b - \lambda^u\right)\left[y(k^*) - k^*\right]}{1-\beta[1-\delta-\alpha(\underline{\theta})]}\ell_1 \\ &\quad - \frac{\zeta\underline{\theta} + \beta[1-\delta-\alpha(\underline{\theta})]\left(\lambda^b - \lambda^u\right)\left[y(k^*) - k^*\right]}{1-\beta[1-\delta-\alpha(\underline{\theta})]}\underline{\ell} \\ &\quad + \frac{-(1-\underline{\ell})\zeta\underline{\theta} + \beta\lambda^u\left[y\left(k^*\right) - k^*\right] + \beta\underline{\ell}\left(\lambda^b - \lambda^u\right)\left[y(k^*) - k^*\right]}{1-\beta},\end{aligned} \tag{67}$$

where we obtained the second equality by substituting $\mathbb{W}_1^0$ by its expression given by (66) and we used (59) to obtain

$$\begin{aligned}\mathcal{W}_1 &= -(1+\rho)\zeta(1-\ell_0)\theta_0 + (1-\ell_1)\lambda^u\left[y\left(k_1^u\right) - k_1^u\right] \\ &\quad + \ell_1\lambda^b\left[y(k_1^b) - k_1^b - \psi\left(k_1^b - m_1^b\right)\right].\end{aligned}$$

We differentiate $\tilde{\mathbb{W}}_0 = (1+\rho)\mathbb{W}_0$ given by (67) with respect to s_1:

$$\begin{aligned}\frac{\partial \tilde{\mathbb{W}}_0}{\partial s_1} &= \frac{\partial \tilde{\mathbb{W}}_0}{\partial \theta_0}\frac{\partial \theta_0}{\partial s_1} + (1-\ell_1)\lambda^u \left[y'\left(k_1^u\right) - 1\right]\frac{\partial k_1^u}{\partial s_1} \\ &\quad + \ell_1 \lambda^b \psi'\left(k_1^b - m_1^b\right)\frac{\partial m_1^b}{\partial s_1}, \end{aligned} \tag{68}$$

where we used that $\partial \tilde{\mathbb{W}}_0/\partial k_1^u = (1-\ell_1)\lambda^u\left[y'\left(k_1^u\right)-1\right]$, $\partial \tilde{\mathbb{W}}_0/\partial m_1^b = \ell_1\lambda^b\psi'\left(k_1^b - m_1^b\right)$. The derivative on the right side of (68) is equal to

$$\begin{aligned}(1-\ell_0)^{-1}\frac{\partial \tilde{\mathbb{W}}_0}{\partial \theta_0} &= -(1+\rho)\zeta - \alpha'(\theta_0)\lambda^u\left[y\left(k_1^u\right) - k_1^u\right] \\ &\quad + \alpha'(\theta_0)\lambda^b\left[y(k_1^b) - k_1^b - \psi\left(k_1^b - m_1^b\right)\right] \\ &\quad + \alpha'(\theta_0)\frac{\zeta\underline{\theta} + \beta[1-\delta-\alpha(\underline{\theta})]\left(\lambda^b - \lambda^u\right)\left[y(k^*) - k^*\right]}{1-\beta[1-\delta-\alpha(\underline{\theta})]},\end{aligned}$$

where we used, from (61), $\partial \ell_1/\partial\theta_0 = \alpha'(\theta_0)(1-\ell_0)$. From (21) credit market tightness at $t=0$ solves:

$$\frac{\theta_0}{\alpha(\theta_0)} = \frac{\beta\eta\left[\Delta\pi(s_1)\right]}{\zeta} - \beta(1-\eta)\underline{\theta} + \beta(1-\delta)\frac{\underline{\theta}}{\alpha(\underline{\theta})}. \tag{69}$$

By differentiating (69) and using that $\partial\Delta\pi(s_1)/\partial s_1 = \left(m_1^u - m_1^b\right)$, we obtain:

$$\frac{\partial\theta_0}{\partial s_1} = \frac{\alpha(\theta_0)}{1-\epsilon(\theta_0)}\frac{\beta\eta(m_1^u - m_1^b)}{\zeta}. \tag{70}$$

Differentiating (12) and (15) assuming $m_1^b > 0$,

$$\frac{\partial m_1^u}{\partial s_1} = \frac{1}{\lambda^u y''(m_1^u)} \tag{71}$$

$$\frac{\partial m_1^b}{\partial s_1} = \frac{\psi''\left(k_1^b - m_1^b\right) - y''(k_1^b)}{\psi''\left(k_1^b - m_1^b\right)\lambda^b y''(k_1^b)}. \tag{72}$$

Substituting (70), (71), and (72) into $\partial\mathbb{W}_0/\partial s_1$ and rearranging, we obtain:

$$\begin{aligned}\frac{\partial \tilde{\mathbb{W}}_0}{\partial s_1} &= \frac{\partial \tilde{\mathbb{W}}_0}{\partial \theta_0}\frac{\alpha(\theta_0)}{1-\epsilon(\theta_0)}\frac{\beta\eta(m_1^u - m_1^b)}{\zeta} \\ &\quad + (1-\ell_1)\frac{\left[y'\left(k_1^u\right) - 1\right]}{y''(m_1^u)} \\ &\quad + \ell_1\psi'\left(k_1^b - m_1^b\right)\frac{\psi''\left(k_1^b - m_1^b\right) - y''(k_1^b)}{\psi''\left(k_1^b - m_1^b\right)y''(k_1^b)}.\end{aligned}$$

Dividing by $\left(m_1^u - m_1^b\right)$ and taking the limit as s_1 approaches 0:

$$\begin{aligned}\lim_{s_1\to 0}\left(\frac{1}{\left(m_1^u - m_1^b\right)}\frac{\partial \tilde{\mathbb{W}}_0}{\partial s_1}\right) &= \frac{\partial \tilde{\mathbb{W}}_0}{\partial \theta_0}\frac{\alpha(\underline{\theta})}{1-\epsilon(\underline{\theta})}\frac{\beta\eta}{\zeta} \\ &\quad + (1-\ell_1)\frac{\lambda^b\psi''(0)}{\left(\lambda^b - \lambda^u\right)\psi''(0) + \lambda^u y''\left(k^*\right)} \\ &\quad + \ell_1\frac{\lambda^u}{\left(\lambda^b - \lambda^u\right)\psi''(0) + \lambda^u y''\left(k^*\right)}\frac{\psi''(0) - y''(k^*)}{\xi(0)}.\end{aligned} \tag{73}$$

In order to obtain the second term on the right side, we used that

$$\frac{y'\left(k_1^u\right)-1}{\left(m_1^u-m_1^b\right)}=\frac{s_1}{\lambda^u\left[y'^{-1}\left(1+s_1/\lambda^u\right)-y'^{-1}\left(1+s_1/\lambda^b\right)+\psi'^{-1}(s_1/\lambda^b)\right]}.$$

Applying L'Hôpital's Rule and multiplying by $1/y''(k^*)$, the term on the right side approaches to

$$\frac{\lambda^b\psi''(0)}{\left(\lambda^b-\lambda^u\right)\psi''(0)+\lambda^u y''\left(k^*\right)}.$$

In order to obtain the third term on the right side we used that

$$\ell_1\frac{k_1^b-m_1^b}{m_1^u-m_1^b}\frac{\psi'\left(k_1^b-m_1^b\right)}{\left(k_1^b-m_1^b\right)\psi''\left(k_1^b-m_1^b\right)}\frac{\psi''\left(k_1^b-m_1^b\right)-y''(k_1^b)}{y''(k_1^b)}.$$

Moreover,

$$\frac{k_1^b-m_1^b}{m_1^u-m_1^b}=\frac{\psi'^{-1}(s_1/\lambda^b)}{y'^{-1}\left(1+s_1/\lambda^u\right)-y'^{-1}\left(1+s_1/\lambda^b\right)+\psi'^{-1}(s_1/\lambda^b)},$$

which, by L'Hôpital's Rule, tends to

$$\frac{\lambda^u y''\left(k^*\right)}{\left(\lambda^b-\lambda^u\right)\psi''(0)+\lambda^u y''\left(k^*\right)}.$$

Finally,

$$(1-\ell_0)^{-1}\left.\frac{\partial\tilde{\mathbb{W}}_0}{\partial\theta_0}\right|_{s_1=0}=-(1+\rho)\zeta+\alpha'(\underline{\theta})\frac{\left(\lambda^b-\lambda^u\right)\left[y(k^*)-k^*\right]+\zeta\underline{\theta}}{1-\beta[1-\delta-\alpha(\underline{\theta})]}. \tag{74}$$

From the free-entry condition (69) when $s_1=0$:

$$\frac{(1+\rho)\underline{\theta}\zeta}{\eta\alpha(\underline{\theta})}=\frac{\left(\lambda^b-\lambda^u\right)\left[y(k^*)-k^*\right]+\zeta\underline{\theta}}{1-\beta[1-\delta-\alpha(\underline{\theta})]}, \tag{75}$$

where we used that $\theta_0=\underline{\theta}$ and $\Delta\pi(0)=\left(\lambda^b-\lambda^u\right)\left[y(k^*)-k^*\right]$. Substituting (75) into (74):

$$(1-\ell_0)^{-1}\left.\frac{\partial\tilde{\mathbb{W}}_0}{\partial\theta_0}\right|_{s_1=0}=(1+\rho)\zeta\left[\frac{\epsilon(\underline{\theta})-\eta}{\eta}\right].$$

Substituting this expression into (73), $s_1>0$ is optimal if:

$$\begin{aligned}&(1-\ell_0)\left[\epsilon(\underline{\theta})-\eta\right]\frac{\alpha(\underline{\theta})}{1-\epsilon(\underline{\theta})}+(1-\ell_1)\frac{\lambda^b\psi''(0)}{\left(\lambda^b-\lambda^u\right)\psi''(0)+\lambda^u y''\left(k^*\right)}\\&+\ell_1\frac{\lambda^u}{\left(\lambda^b-\lambda^u\right)\psi''(0)+\lambda^u y''\left(k^*\right)}\frac{\psi''\left(0\right)-y''(k^*)}{\xi(0)}>0,\end{aligned}$$

where, from (61), $\ell_1=(1-\delta)\ell_0+\alpha(\underline{\theta})(1-\ell_0)$. Assume $\psi(\mathrm{L})=\mathrm{L}^{1+\xi}/(1+\xi)$ with $\xi>1$. The condition above can be rewritten as:

$$\epsilon(\underline{\theta})-\eta>\frac{1-\epsilon(\underline{\theta})}{\alpha(\underline{\theta})}\left(\frac{\ell_1}{1-\ell_0}\frac{1}{\xi}\right).$$

Plugging ℓ_1 by its expression we obtain (58). ■

Proposition 7 ***(Ramsey problem.)*** *The policymaker's value function solves*

$$\mathbb{W}(\ell_0) = \max_{\theta_0 \in \Omega = \left[\underline{\theta}, \bar{\theta}\right]} \widetilde{\mathbb{W}}(\ell_0, \theta_0), \tag{76}$$

where $\widetilde{\mathbb{W}}$ *is the unique solution in* $\mathcal{B}([0,1] \times \Omega)$ *to (26) where*

$$\underline{\theta} = \frac{\frac{\bar{\alpha}\beta\eta}{\zeta}\left(\lambda^b - \lambda^u\right)[y(k^*) - k^*] + \beta(1-\delta) - 1}{1 - \beta\left[1 - \delta - \bar{\alpha}(1-\eta)\right]} \tag{77}$$

$$\bar{\theta} = \frac{\bar{\alpha}\beta\eta\lambda^b\left[y(\hat{k}) - \hat{k} - \psi(\hat{k})\right]/\zeta + \beta(1-\delta) - 1}{1 - \beta\left[1 - \delta - \bar{\alpha}(1-\eta)\right]}. \tag{78}$$

Proof of Proposition 7. First, we characterize the state space. Given the functional form $\alpha(\theta) = \bar{\alpha}\theta/(1+\theta)$ and the parametric condition $\delta + \bar{\alpha}(1-\eta) < 1$, the law of motion for market tightness, (21), can be rewritten as

$$\theta_t = \frac{\bar{\alpha}\beta\eta}{\zeta}\Delta\pi(s_{t+1}) + \beta\left[1 - \delta - \bar{\alpha}(1-\eta)\right]\theta_{t+1} + \beta(1-\delta) - 1. \tag{79}$$

We restrict the policymaker's choice to bounded sequences $\{\theta_t\}$ that solve (79) given some initial condition, θ_0. The set of values for market tightness, Ω, is obtained as follows. We define $\bar{\theta}$ as steady-state credit market tightness when $s = \infty$, in which case $\Delta\pi(+\infty) = \lambda^b\left[y(\hat{k}) - \hat{k} - \psi(\hat{k})\right]$. From (79) it solves:

$$\bar{\theta} = \frac{\bar{\alpha}\beta\eta}{\zeta}\Delta\pi(+\infty) + \beta\left[1 - \delta - \bar{\alpha}(1-\eta)\right]\bar{\theta} + \beta(1-\delta) - 1.$$

Solving for $\bar{\theta}$, we obtain (78), i.e.,

$$\bar{\theta} = \frac{\bar{\alpha}\beta\eta\lambda^b\left[y(\hat{k}) - \hat{k} - \psi(\hat{k})\right]/\zeta + \beta(1-\delta) - 1}{1 - \beta\left[1 - \delta - \bar{\alpha}(1-\eta)\right]}.$$

Suppose $\theta_t > \bar{\theta}$ for some t. From (79),

$$\theta_{t+1} - \bar{\theta} = \frac{\theta_t - \bar{\theta} - \bar{\alpha}\beta\eta\left\{\Delta\pi(s_{t+1}) - \Delta\pi(+\infty)\right\}/\zeta}{\beta\left[1 - \delta - \bar{\alpha}(1-\eta)\right]}.$$

For all $s_{t+1} \in [0, \infty)$, $\Delta\pi(s_{t+1}) - \Delta\pi(\infty) \leq 0$. Since $\beta\left[1 - \delta - \bar{\alpha}(1-\eta)\right] \in (0,1)$, the sequence $\{\theta_t - \bar{\theta}\}$ is increasing and unbounded, which is inconsistent with an equilibrium. Next, we define $\underline{\theta}$ as steady-state market tightness when $s = 0$, in which case $\Delta\pi(0) = \left(\lambda^b - \lambda^u\right)[y(k^*) - k^*]$. From (79), it solves:

$$\underline{\theta} = \frac{\bar{\alpha}\beta\eta}{\zeta}\Delta\pi(0) + \beta\left[1 - \delta - \bar{\alpha}(1-\eta)\right]\underline{\theta} + \beta(1-\delta) - 1.$$

Solving for $\underline{\theta}$, we obtain (77), i.e.,

$$\underline{\theta} = \frac{\bar{\alpha}\beta\eta\left(\lambda^b - \lambda^u\right)[y(k^*) - k^*]/\zeta + \beta(1-\delta) - 1}{1 - \beta\left[1 - \delta - \bar{\alpha}(1-\eta)\right]}.$$

Suppose $\theta_t \in (0, \underline{\theta})$ for some t. With $\underline{\theta} > 0$, from (79),

$$\theta_{t+1} - \underline{\theta} = \frac{\theta_t - \underline{\theta} - \bar{\alpha}\beta\eta\left\{\Delta\pi(s_{t+1}) - \Delta\pi(0)\right\}/\zeta}{\beta\left[1 - \delta - \bar{\alpha}(1-\eta)\right]}.$$

For all $s_{t+1} \in [0, +\infty)$, $\Delta\pi(s_{t+1}) - \Delta\pi(0) \geq 0$. So θ_t becomes negative in finite time, which is inconsistent with an equilibrium. Finally, for all $\theta_t \in [\underline{\theta}, \bar{\theta}]$ there exists a θ_{t+1} consistent with an equilibrium, e.g., a steady-state path where $\theta_{t+\tau} = \theta_t$ for all $\tau > 1$.

The feasibility condition $\theta_{t+1} \in \Gamma(\theta_t)$ is obtained from (79) by varying s_{t+1} from 0 to $+\infty$, i.e.,

$$\Gamma(\theta_t) = \left[\frac{\theta_t - \bar{\alpha}\beta\eta\left(\lambda^b - \lambda^u\right)[y(k^*) - k^*]/\zeta - \beta(1-\delta) + 1}{\beta[1-\delta-\bar{\alpha}(1-\eta)]}, \right. \tag{80}$$

$$\left. \frac{\theta_t - \bar{\alpha}\beta\eta\lambda^b\left[y(\hat{k}) - \hat{k} - \psi(\hat{k})\right]/\zeta - \beta(1-\delta) + 1}{\beta[1-\delta-\bar{\alpha}(1-\eta)]} \right] \cap \Omega.$$

Given a $\theta_{t+1} \in \Gamma(\theta_t)$, the remaining choice variables of the planner are determined recursively according to:

$$\Delta\pi(s_{t+1}) = \frac{\zeta\left\{\theta_t - \beta[1-\delta-\bar{\alpha}(1-\eta)]\theta_{t+1} - \beta(1-\delta) + 1\right\}}{\bar{\alpha}\beta\eta} \tag{81}$$

$$m^u_{t+1} = y'^{-1}\left(1 + \frac{s_{t+1}}{\lambda^u}\right) \tag{82}$$

$$k^b_{t+1} = \max\left\{y'^{-1}\left(1 + \frac{s_{t+1}}{\lambda^b}\right), \hat{k}\right\} \tag{83}$$

$$\mathrm{L}_{t+1} = \psi'^{-1}\left[y'(k^b_{t+1}) - 1\right] \tag{84}$$

$$\ell_{t+1} = (1-\delta)\ell_t + \frac{\bar{\alpha}\theta_t}{1+\theta_t}(1-\ell_t). \tag{85}$$

We now turn to the Bellman equation (26). For a given initial market tightness, θ_0, we can apply the Principle of Optimality to show the value function of the planner, $\widetilde{\mathbb{W}}(\ell_t, \theta_t)$, solves the Bellman equation (26). It is the fixed point of a mapping from $\mathcal{B}([0,1] \times [\underline{\theta}, \bar{\theta}])$ into itself. The mapping in (26) is a contraction by Blackwell's sufficient conditions (Theorem 3.3 in Stokey and Lucas, 1989), and by the contraction mapping theorem (Theorem 3.2 in Stokey and Lucas 1989), the fixed point exists and is unique. The correspondence Γ is continuous and the policymaker's period utility is also continuous. So $\widetilde{\mathbb{W}}(\ell, \theta)$ is continuous by the Contraction Mapping Theorem. Given there is no initial value for θ in the original sequence problem, (27), $\theta_0 \in \Omega$ is chosen to as to maximize $\widetilde{\mathbb{W}}(\ell_0, \theta_0)$. Such a solution exists by the continuity of $\widetilde{\mathbb{W}}(\ell_0, \theta_0)$ and the compactness of Ω. ■

Appendix A2: Continuous time limit

We now derive the continuous-time limit of our model (see Choi and Rocheteau 2020 for a detailed presentation of New Monetarist models in continuous time). Let Δ denote the length of a period of time, where Δ is assumed to be small. We rewrite all variables that have a time dimension as being proportional to Δ. It includes the matching function, the separation rate, the rate of time preference, and the real return on liquid assets.

The law of motion of the lending relationships is

$$\ell_{t+\Delta} = (1-\delta\Delta)\ell_t + \alpha(\theta_t)\Delta(1-\ell_t).$$

We subtract ℓ_t on both sides, divide by Δ, and take the limit as Δ goes to zero to obtain

$$\dot{\ell}_t = \alpha(\theta_t)(1-\ell_t) - \delta\ell_t,$$

where $\dot{\ell}_t = \lim_{\Delta\to 0}\left(\ell_{t+\Delta} - \ell_t\right)/\Delta$.

The profits of an unbanked entrepreneur are $\pi_t^u\Delta$ where

$$\pi_t^u = \max_{m_t\geq 0}\left\{-s_t m_t + \lambda^u \max_{k_t\leq m_t}\left[y(k_t) - k_t\right]\right\},$$

where the interest spread is

$$s_t\Delta = \frac{(\rho - r_t)\,\Delta}{1 + r_t\Delta}.$$

Note that the profits conditional on an investment opportunity, $y(k_t) - k_t$, is a stock that has no time dimension. Taking the limit as Δ goes to zero,

$$s_t = \rho - r_t.$$

The first-order condition gives

$$s_t = \lambda^u\left[y'(m_t^u) - 1\right].$$

The flow profits of banked entrepreneurs are unaffected,

$$\pi^b(s_t) = \max_{k^b,m^b\geq 0}\left\{\lambda^b\left[y(k^b) - k^b - \psi(k^b - m^b)\right] - s_t m^b\right\}.$$

The first-order conditions are

$$\psi'\left(k_t^b - m_t^b\right) = y'(k_t^b) - 1 \leq \frac{s_t}{\lambda^b}, \quad \text{"="} \text{ if } m_t^b > 0, \quad \forall t.$$

Finally, the free-entry condition for banks is

$$(1+\rho\Delta)\frac{\theta_t}{\alpha(\theta_t)\Delta} = \frac{\eta\left[\pi^b(s_{t+\Delta}) - \pi^u\left(s_{t+\Delta}\right)\right]\Delta}{\zeta\Delta} - (1-\eta)\theta_{t+\Delta} + (1-\delta\Delta)\frac{\theta_{t+\Delta}}{\alpha(\theta_{t+\Delta})\Delta}.$$

Notice the cost of bank entry is a flow cost, and hence proportional to Δ. Rearranging, the equation can be rewritten as:

$$\frac{1}{\Delta}\left(\frac{\theta_t}{\alpha(\theta_t)} - \frac{\theta_{t+\Delta}}{\alpha(\theta_{t+\Delta})}\right) + \rho\frac{\theta_t}{\alpha(\theta_t)} + \delta\frac{\theta_{t+\Delta}}{\alpha(\theta_{t+\Delta})} = \frac{\eta\left[\pi^b(s_{t+\Delta}) - \pi^u\left(s_{t+\Delta}\right)\right]}{\zeta} - (1-\eta)\theta_{t+\Delta}.$$

Taking the limit as Δ goes to zero,

$$-\left(\frac{1-\epsilon(\theta_t)}{\alpha(\theta_t)}\right)\dot{\theta} + (\rho+\delta)\frac{\theta_t}{\alpha(\theta_t)} = \frac{\eta\left[\pi^b(s_t) - \pi^u\left(s_t\right)\right]}{\zeta} - (1-\eta)\theta_t,$$

where $\epsilon(\theta) \equiv \theta\alpha'(\theta)/\alpha(\theta)$. Rearranging, we obtain:

$$\dot{\theta}_t = \frac{\alpha(\theta_t)}{1-\epsilon(\theta_t)}\left\{(\rho+\delta)\frac{\theta_t}{\alpha(\theta_t)} + (1-\eta)\theta_t - \frac{\eta\left[\pi^b(s_t) - \pi^u(s_t)\right]}{\zeta}\right\}.$$

To summarize, the system of ODEs for the measure of lending relationships, credit market tightness, and the market-clearing condition for liquid assets are

$$\dot{\ell}_t = \alpha(\theta_t)(1-\ell_t) - \delta\ell_t \tag{86}$$

$$\dot{\theta}_t = \frac{\alpha(\theta_t)}{1-\epsilon(\theta_t)}\left\{(\rho+\delta)\frac{\theta_t}{\alpha(\theta_t)} + (1-\eta)\theta_t - \frac{\eta\Delta\pi(s_t)}{\zeta}\right\} \tag{87}$$

$$\mathrm{M}_t = \ell_t m^b(s_t) + (1-\ell_t)m^u(s_t), \tag{88}$$

where $m^u(s_t)$ and $m^b(s_t)$ are the implicit solutions to (12) and (15) with $s_t \equiv \rho - r_t$, and M_t is the supply of liquid assets. An equilibrium is a time path, $\{\ell_t, \theta_t, \mathrm{M}_t, s_t\}$, that solves (86)-(88) given ℓ_0 and monetary policy formulated either as s_t or M_t.

If s is kept constant then the equilibrium can be solved recursively. The time path for θ_t is obtained from (87). In the neighborhood of $\dot{\theta}_t = 0$, $\partial\dot{\theta}_t/\partial\theta_t > 0$. Hence, the unique solution leading to the steady state is $\theta_t = \theta^s$ for all t. Given θ_t and ℓ_0 we can solve for ℓ_t from (86). Given ℓ_t we can solve for M_t from (88).

Suppose next that M_t is kept constant at $\mathrm{M} < k^*$. From (88) we can express the market clearing spread as $s(\ell, \mathrm{M})$. From the observation that $m^b(s)$ and $m^u(s)$ are decreasing functions of s, it follows that s is decreasing in M. Under the assumption that $m^b(s) < m^u(s)$, an increase in ℓ reduces the aggregate demand for liquid assets. In order to restore market clearing, the spread must fall. Hence, s is decreasing in ℓ. Substituting $s(\ell,\mathrm{M})$ into (87) we can reduce an equilibrium to a pair of time paths, (ℓ_t, θ_t), solution to a system of autonomous ODEs:

$$\dot{\ell}_t = \alpha(\theta_t)(1-\ell_t) - \delta\ell_t \tag{89}$$

$$\dot{\theta}_t = \frac{\alpha(\theta_t)}{1-\epsilon(\theta_t)}\left\{(\rho+\delta)\frac{\theta_t}{\alpha(\theta_t)} + (1-\eta)\theta_t - \frac{\eta\Delta\pi\left[s(\ell,\mathrm{M})\right]}{\zeta}\right\} \tag{90}$$

From (89) the ℓ-isocline is upward-sloping while from (90) the θ-isocline is downward-sloping. The signs of the terms of the Jacobian matrix in the neighborhood of the steady state are:

$$J = \begin{pmatrix} - & + \\ + & + \end{pmatrix}.$$

It follows that $\det J < 0$, i.e., the steady state is a saddle point and there is a unique saddle path leading to it.

Appendix A3: Search for investment opportunities

We now endogenize the probabilities of investment opportunities, λ^u and λ^b. We assume that both the entrepreneur and the bank can exert some effort in stage 1, e^f and e^b, respectively, to search for profit opportunities. The disutility of effort for both agents is $-e$. The probability of an investment opportunity, $\Lambda(E)$, is a function of the joint effective effort, denoted E, with $\Lambda(0) = 0$, $\Lambda' > 0$, $\Lambda'' < 0$, $\Lambda'(0) = +\infty$, $\Lambda'(+\infty) = 0$ and

$$E(e^f, e^b) = \left[(e^f)^\varepsilon + \kappa (e^b)^\varepsilon \right]^{\frac{1}{\varepsilon}}, \tag{91}$$

where $\varepsilon \in (0,1)$. The joint effective effort, E, combines the individual efforts of the entrepreneur and the bank according to a CES technology. If $\varepsilon = 1$ the efforts of the entrepreneurs and the bank are perfect substitutes. Moreover, if $\kappa < 1$, entrepreneurs are more efficient at generating profit opportunities than banks are. If $\varepsilon < 1$ then the individual efforts are imperfect substitutes, which means that there are investment opportunities that the bank can bring that could not be brought by the entrepreneur alone. For instance, according to the FDIC Small Business Lending Survey (2018, p.6), "small banks are regarded as having a comparative advantage in gathering and using "soft" information—knowledge of both the local community and the small businesses within it—which the bank has accumulated over multiple interactions, and which is hard to quantify or transmit". If $\kappa = 0$, then only entrepreneurs contribute to the arrival of investment opportunities. If $\kappa > 0$ then banks contributes positively to the arrival of profit opportunities.

The problem of the unbanked entrepreneur becomes

$$\pi^u(s_t) \equiv \max_{m_t \geq 0, e_t^f \geq 0} \left\{ -s_t m_t - e_t^f + \Lambda(e_t^f) \max_{k_t \leq m_t} \left[y(k_t) - k_t \right] \right\}. \tag{92}$$

When the entrepreneur is unmatched, $E^u = e^f$. The first-order condition with respect to e^u gives

$$1 = \Lambda'(E_t^u) \left[y(k_t^u) - k_t^u \right]. \tag{93}$$

It can be checked that E_t^u and m_t are complements and both decrease with s_t. The investment probability is now endogenous, $\lambda^u = \Lambda(E^u)$, and it decreases with s_t.

The joint profits of a lending relationship solve

$$\pi^b(s_t) = \max_{k^b, m^b, e^f, e^b} \left\{ -s_t m^b - (e_t^f + e_t^b) + \Lambda \left[E(e_t^f, e_t^b) \right] \left[y(k^b) - k^b - \psi(k^b - m^b) \right] \right\}. \tag{94}$$

The bank and the entrepreneur coordinate their search efforts to maximize the joint profits. The first-order conditions with respect to e_t^f and e_t^b are:

$$\frac{1}{\kappa} \left(\frac{e^b}{E} \right)^{1-\varepsilon} = \left(\frac{e^f}{E} \right)^{1-\varepsilon} = \Lambda'(E) \left[y(k^b) - k^b - \psi(k^b - m^b) \right]. \tag{95}$$

After some manipulations, $e^b = \kappa^{\frac{1}{1-\varepsilon}} e^f$. The search effort of the bank is proportional to the one of the entrepreneur where the coefficient of proportionality is $\kappa^{\frac{1}{1-\varepsilon}}$. Provided that $\kappa \in (0,1)$, it is optimal for the bank to search for profit opportunities, but the bank's effort is less than the one of the entrepreneur. Moreover, if $\kappa < 1$ and ε tends to 1, then e^b approaches 0. The aggregate effort is equal to $E^b = \left(1 + \kappa^{\frac{1}{1-\varepsilon}} \right)^{\frac{1}{\varepsilon}} e^f$ and E^b solves

$$\left(1 + \kappa^{\frac{1}{1-\varepsilon}} \right)^{\frac{\varepsilon - 1}{\varepsilon}} = \Lambda' \left(E^b \right) \left[y(k^b) - k^b - \psi(k^b - m^b) \right]. \tag{96}$$

The investment probability is $\lambda^b = \Lambda(E^b)$. We now obtain the following lemma regarding the ranking of λ^b and λ^u at the Friedman rule.

Proposition 8 ***(Search for investment opportunities at the Friedman rule.)*** *Suppose $s = 0$. If $\varepsilon = 1$ then $\lambda^b = \lambda^u$. If $\varepsilon \in (0, 1)$ then $\lambda^b > \lambda^u$.*

Proof. Consider first the case $\varepsilon = 1$. The FOCs are:

$$\begin{aligned} -\frac{1}{\kappa} + \Lambda'\left(E^b\right)\left[y(k^b) - k^b - \psi(k^b - m^b)\right] &\leq 0, \text{ " = " if } e^b > 0 \\ -1 + \Lambda'\left(E^b\right)\left[y(k^b) - k^b - \psi(k^b - m^b)\right] &\leq 0, \text{ " = " if } e^f > 0. \end{aligned}$$

If $\kappa < 1$, it follows immediately that $e^b = 0$ and $e^f > 0$. When $s = 0$, $k^b = m^b$ and the FOC simplifies to:

$$-1 + \Lambda'\left(E^b\right)\left[y(k^*) - k^*\right] = 0.$$

The comparison with (93) shows that $E^u = E^b$ and $\lambda^u = \lambda^b$. Consider the case where $\varepsilon \in (0, 1)$. From (96), at the Friedman rule, $s = 0$, the joint search effort in a relationship solves:

$$\Lambda'\left(E^b\right) = \frac{\left(1 + \kappa^{\frac{1}{1-\varepsilon}}\right)^{\frac{\varepsilon - 1}{\varepsilon}}}{y(k^*) - k^*}.$$

From (93), the search effort of an unbanked entrepreneur, E^u, solves

$$\Lambda'(E^u) = \frac{1}{y(k^*) - k^*}.$$

For all $\kappa \in (0, 1)$, $\left(1 + \kappa^{\frac{1}{1-\varepsilon}}\right)^{\frac{\varepsilon - 1}{\varepsilon}} < 1$. Hence, $\Lambda'\left(E^b\right) < \Lambda'(E^u)$, which implies $E^b > E^u$ and $\lambda^b \equiv \Lambda\left(E^b\right) > \lambda^u \equiv \Lambda\left(E^u\right)$. ■

According to Lemma 4, if the search efforts of the bank and the entrepreneur are perfect substitutes, then λ^u and λ^b are equal at the Friedman rule. However, as long as those efforts are not perfect substitutes, the investment probability of a banked entrepreneur is larger than the one of an unbanked entrepreneur at the Friedman rule, $\lambda^b > \lambda^u$.

We now turn to the optimal monetary policy. We set $\varepsilon = 0.5$ and $\Lambda(e) = \min\{\gamma\sqrt{e}, 1\}$. We calibrate (γ, κ) such that the endogenous rate of investment opportunities for banked and unbanked firms matches the parameters from the baseline calibration in steady state, $\Lambda(E^u) = \lambda^u$ and $\Lambda(E^b) = \lambda^b$. This ensures we match the same moments as our baseline for the elasticity of liquidity demand by unbanked firms relative to the user cost and the impact of relationships on investment opportunities. This procedure yields $\gamma = 0.16$ and $\kappa = 0.41$. We also recalibrate other parameters $(\overline{\alpha}, \eta, \xi, B, \zeta)$ following the baseline strategy and report these in the following table. The parameters remain little changed with the exception of entry costs, ζ, which fall in order to compensate for the added costs of effort.

Specification	ρ	ξ	δ	$\bar{\alpha}$	λ^u	λ^b	B	η	ζ
Baseline	0.0033	15.7	0.012	0.465	0.029	0.034	3.42	0.08	0.015
Endogenous λ	0.0033	15.7	0.012	0.483	0.029	0.034	3.42	0.06	0.005

We measure social welfare from stage 2 of period t until stage 1 of period $t + 1$ by

$$\begin{aligned} &-\zeta\theta_t(1 - \ell_t) + \beta(1 - \ell_{t+1})\left\{\Lambda(e^u_{t+1})\left[y(m^u_{t+1}) - m^u_{t+1}\right] - e^u_{t+1}\right\} \\ &+\beta\ell_{t+1}\left\{\Lambda(E^b_{t+1})\left[y\left(k^b_{t+1}\right) - k^b_{t+1} - \psi(\mathrm{L}_{t+1})\right] - e^f_{t+1} - e^b_{t+1}\right\}. \end{aligned}$$

So the search efforts enter directly into the welfare function. Figure 8 illustrates the optimal policy response to a 60% destruction of lending relationships. The blue-solid lines reproduce the baseline responses while the dashed-red lines illustrate the optimal policy when investment opportunities are endogenous. The top-three panels how the optimal policy with commitment and the bottom three show optimal policy without. When investment opportunities are endogenous the planner understands that there is an additional margin when setting a path of spreads. Increasing spreads reduces effort and lowers investment opportunities for both banked and unbanked firms. Also, under our calibration, increasing spreads acts to reduce the relative benefit of a lending relationship (see Figure 7). We find that qualitatively these tradeoffs leave the optimal policy unchanged. Quantitatively, the policymaker sets spreads higher when investment opportunities are endogenous compared to when they are exogenous (middle panels in 8).

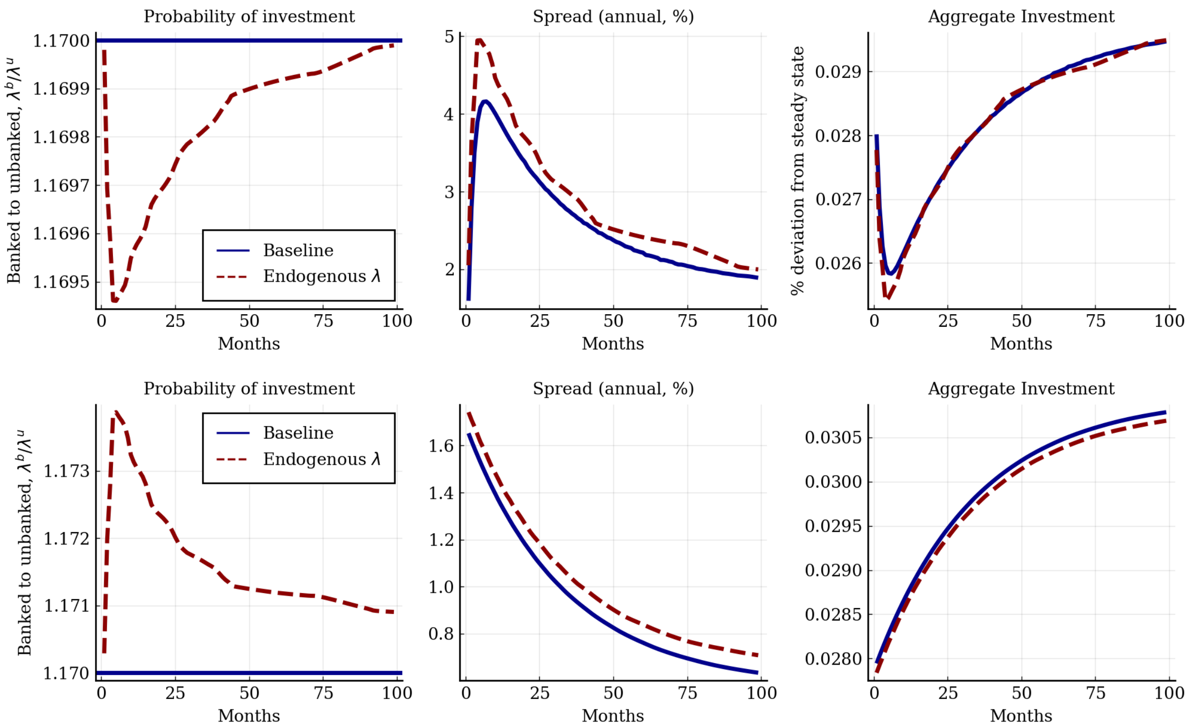

Figure 8: Optimal policy responses with commitment (top three panels) and without commitment (bottom three panels) under endgoenous λ.

Appendix A4. Numerical procedures for optimal policy problems

Optimal policy with commitment

We solve for social welfare $\tilde{\mathbb{W}}(\ell,\theta)$ and the optimal policy function $\theta' = g_\theta(\ell,\theta)$ using value function iteration on a discrete grid.

1. Discretize the state space into $N_\ell \subseteq [0,1]$ and $N_\theta \subseteq \Omega$, where the lower and upper limits of Ω are given by solving for $\theta_t = \theta_{t+1} = \theta$ in 21 when s is either $s = 0$ or $s \to \infty$.

2. Define the Bellman operator as $T\tilde{\mathbb{W}}_n$ given by the right-hand side of (29).

3. Set the initial guess of $\tilde{\mathbb{W}}_0$ given by a policy consistent with a constant spread, s, defined as θ' in
$$\theta = \frac{\bar{\alpha}\beta\eta}{\zeta}\Delta\pi\left(s\right) + \beta(1-\delta) - 1 + \beta\left[1-\delta-\bar{\alpha}(1-\eta)\right]\theta'.$$

4. Update $\tilde{\mathbb{W}}_{n+1}(\ell,\theta) = T\tilde{\mathbb{W}}_n(\ell,\theta)$

5. Repeat 4. until $\max_{\ell,\theta}\left\|\tilde{\mathbb{W}}_{n+1}(\ell,\theta) - \tilde{\mathbb{W}}_n(\ell,\theta)\right\| < \epsilon$. This sequence is Cauchy and converges to the unique fixed point of T.

6. Set $\mathbb{W}\left(\ell_0\right) = \max_{\theta\in\Omega}\tilde{\mathbb{W}}(\ell_0,\theta)$.

Optimal policy without commitment

We solve for the optimal policy functions $\Theta(\ell, m^{u'})$ and $\mathcal{K}(\ell)$ using contraction mappings on a discrete grid.

1. Discretize the state space into $N_\ell \subseteq [0,1]$ and $N_m \subseteq [0,k^*]$, where $y'^*) = 1$.

2. Set the initial guess of $\mathcal{K}_0(\ell)$ under a policy consistent with a constant spread, s, or $s = \lambda^u\left[y(\mathcal{K}(\ell)) - 1\right]$.

3. Given $\mathcal{K}_n(\ell)$, for any $n = 0,1,2,\ldots$, compute $\Theta_n(\ell, m^{u'})$ by iterating over the functional equation $T_{\theta,n}\Theta(\ell, m^{u'})$ given by the right-hand side of 28. Repeat until $\max_{\ell,m^{u'}}\left\|\Theta_{j+1}(\ell,m^{u'}) - \Theta_j(\ell,m^{u'})\right\| < \epsilon$. This sequence is Cauchy and converges to the unique fixed point of $T_{\theta,n}$, which is denoted as $\Theta_n(\ell, m^{u'})$.

4. Given $T_{\theta,n}$ from 3., update $\mathcal{K}_{n+1}(\ell)$ by solving the social welfare function $\mathcal{W}_n(\ell)$ using the functional equation $T_{\mathcal{W},n}\mathcal{W}(\ell)$ given by the right-hand side of 30. Repeat until $\max_\ell\|\mathcal{K}_{j+1}(\ell) - \mathcal{K}_j(\ell)\| < \epsilon$.

5. Check if $\|\mathcal{K}_n(\ell) - \mathcal{K}_{n+1}(\ell)\| < \epsilon$. If true, stop. Else, set $\mathcal{K}_n = \mathcal{K}_{n+1}$ and return to 3.

Appendix A5: Ramsey problem under a timeless approach

Notice the Ramsey solution is not bound by any past promises at time $t = 0$, i.e., it is free to select any equilibrium, in our context by choosing any θ_0. To see how this matters, suppose the size of the credit shock approaches zero, i.e., the economy starts at its stationary solution. Letting the policymaker reset its policy is not innocuous, as illustrated in Figure 9. The policymaker reduces s_1 even though there is no exogenous destruction of lending relationships, which paradoxically, generates a small reduction in ℓ.

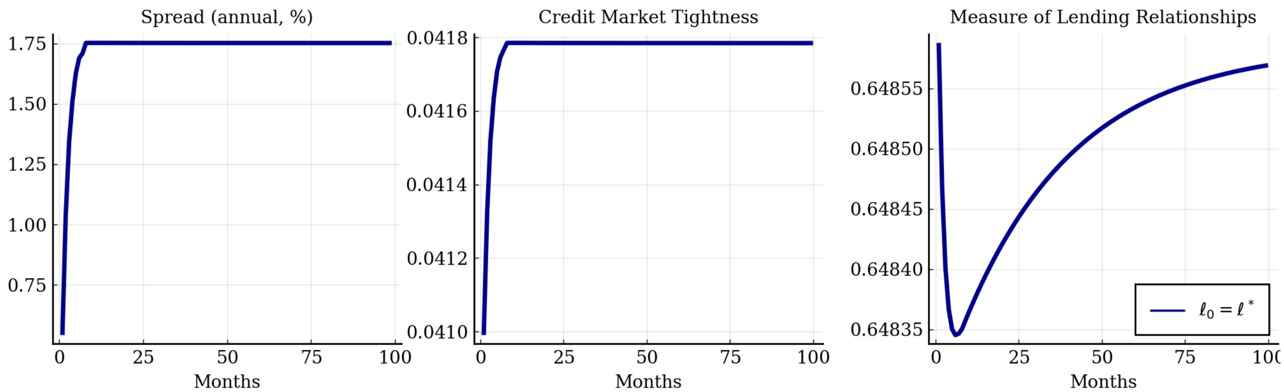

Figure 9: Ramsey solution with reset at $t = 0$ with $\ell_0 = \ell^*$

Woodford (1999, 2003) proposed amending the Ramsey solution to discipline the initial choice of equilibrium. We adopt a similar notion. The solution to the Bellman equation (26) gives a policy function expressed as $\theta_{t+1} = \Theta^*(\ell_t, \theta_t)$. Suppose the economy has an infinite history. Substituting $\theta_t = \Theta^*(\ell_{t-1}, \theta_{t-1})$ and iterating, the policy choice at t can be expressed as a function of the entire history of lending relationships, i.e., $\theta_{t+1} = \Theta^\infty(\ell_t, \ell_{t-1}, \ell_{t-2}, ...)$ or, equivalently, $s_{t+1} = S^\infty(\ell_t, \ell_{t-1}, \ell_{t-2}, ...)$ where S^∞ is a time-invariant policy function that specifies the spread as a function of the entire history of lending relationships.

The Ramsey solution under a timeless approach sets θ_0 such that $\theta_0 = \Theta^\infty(\ell_{-1}, \ell_{-2}, \ell_{-3}, ...)$. If the economy was at a steady state, then $\ell_{-1} = \ell_{-2} = \ell_{-3} = \ell^*$ and $\theta_0 = \Theta^\infty(\ell^*, \ell^*, \ell^*, ...) = \theta^*$. Hence, $\theta_1 = \Theta^*(\ell_0, \theta^*)$ which determines s_1 from the free-entry condition. In particular, if the size of the shock is zero, $\ell_0 = \ell^*$, then $\theta_1 = \Theta^*(\ell^*, \theta^*) = \theta^*$ and the economy remains at its stationary solution.[25] Figure 10 plots the optimal timeless policy outcomes (dashed lines) for different sizes of the shock.

The Ramsey solution under a timeless approach features a hump-shaped path for the spread. By restricting θ_0 to its steady-state value, the timeless approach impacts the initial spread set by the policymaker. For small shocks, s_0 is set close to its long-run value whereas for large shocks, s_0 is close to zero.

[25] As mentioned to us by Edouard Challe, in the case of a one-off, unexpected shock, the timeless perspective does not fit the description of a stationary, fully state-contingent policy plan where the policymaker has planned a long time ago how it would react to any future shock.

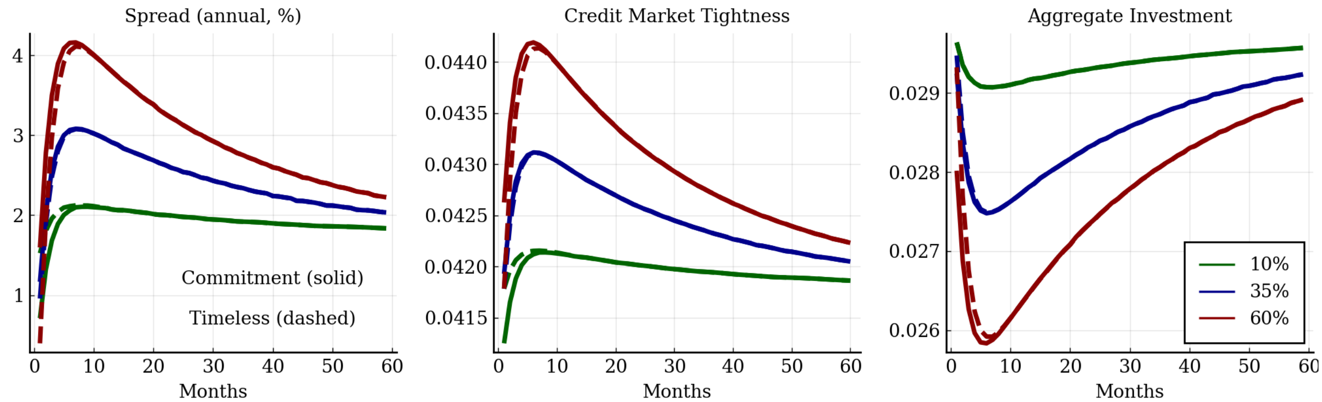

Figure 10: Optimal policy: unrestricted commitment (solid) vs. timeless approach (dashed)

Appendix A6: Calibration details

To capture the impact of a lending relationship on firm innovation, we rely on four empirical studies: Herrera and Minetti (2007), Giannetti (2012), Drexler and Schoar (2014), and Cosci, Meliciani, and Sabato (2016). Below provides a detailed description of how we map the empirical estimates in each of these studies to the implied impact on the rate of investment opportunities, λ^b/λ^u in the model.

Herrera and Minetti (2007) use a 2001 survey of Italian manufacturing firms that gives detailed information on firm/bank relationships and innovation. They find that increasing the length of a banking relationship by 12.5 years increases the probability of product innovation by 94% (an increase of 0.251 relative to 0.267 using Tables 3, 5 and the discussion in the text) and increases the probability of process innovation by 46% (0.204 relative to 0.442). Since we target a shorter duration of lending relationships of 5.6 years, we scale these effects by 5.6/12.5 to get an impact on product innovations of 42% and process innovations of 20.6%. Given product innovation represents 38% of all innovation in the Italian sample (Table 1), gives a weighted effect on innovation of 28%.

Giannetti (2012) also examines the Italian manufacturing firm survey, but uses two waves from 2001 and 2004 to incorporate a panel dimension. Generally, they find a significantly smaller impact of relationship lending to innovation. Relevant for our model, they find that increasing the length of a banking relationship by 8 years increases the probability of product innovation for small firms by 4% (see Table 8 and the discussion in Section 6.2). Scaling by our average relationship duration, as above, implies an effect of (5.6/8)*4% = 2.8%. Further, they find that increasing the number of banks a firm is in a relationship with by one, increases the probability of product innovation by 4.5% (Table 8). Together, these imply an effect of 7.3%. We consider a rough estimate from this study of 5%.

Cosci, Meliciani, and Sabato (2016) use the 2010 Community Innovation Survey of manufacturing firms across European countries (Germany, Spain, France and Italy). They find that a lending relationship (identified by a question that asks if having a long-lasting relationship is the most important factor in a firm's choice of their main bank) increases the probability of product or process innovation by 6.4% (an effect of 0.0415 relative to 0.65, Tables 5 and 1). They also find that increasing the number of lenders by one increases the probability of innovation by 5.3% (an effect of 0.0344 relative to 0.65, Tables 5 and 1). Using both estimates implies an overall effect of 11.7%. Using only the first effect would give 6.4%. We consider the mid-point from this study of 9%.

Finally, Drexler and Schoar (2014) examine the impact of a lending relationship on the probability of credit provision. Using transaction-level data from the major lender to small and medium-sized businesses in Chile, Banco Estado, they find that when a firm's individual loan offer exogenously leaves the relationship that the probability the firm is issued credit falls by 20% (or, in reverse that a lending relationship would increase the probability of credit provision by 23%). They find this effect is predominately driven by credit applications and to a lesser extent by credit approvals. While this study does not give an estimate of the effect on innovation directly, we take consider the probability of credit provision to be in line with the probability of innovation.

While these studies all vary in their identification strategies, controls, etc., they all generally find a positive impact of lending relationships. In our baseline calibration we take the mid-point of these estimates of 17%, but stress that our results go through even when we consider the lower end of the range of 5% and the higher end of the range of 28%.

Appendix A7: Quantitative robustness

In the following, we discuss how optimal monetary policy depends on some key parameters of the model. When applicable, we also report the sensitivity of the calibrated parameters to the empirical moments. We first consider banks' bargaining power, that determines whether there is too much or too little creation of relationships. In our benchmark calibration, $\eta = 0.08$. We report optimal monetary policy when $\eta \in \{0.04, 0.16\}$, double and half of our baseline. This corresponds to using estimates of the net interest margin between 2% and 10%. Our next check considers robustness with respect to the elasticity of the cost of external finance. We take half of the baseline estimate $\xi = 8.0$. This corresponds to increasing the targeted semi-elasticity of m^u/m^b to s from 11.98 to 18.44. Finally, we consider robustness with respect to the difference in investment opportunities between banked and unbanked firms. As discussed in Section A6, we consider the min and max of estimates in the literature on the impact of a lending relationship in firm innovation of $\lambda^b = 1.05\lambda^u$ and $\lambda^b = 1.28\lambda^u$, respectively. In each case, we recalibrate $\bar{\alpha}$ to target a long-run level of lending relationships of 65% and keep the other parameters of the model unchanged. All parameters are reported in the table below.

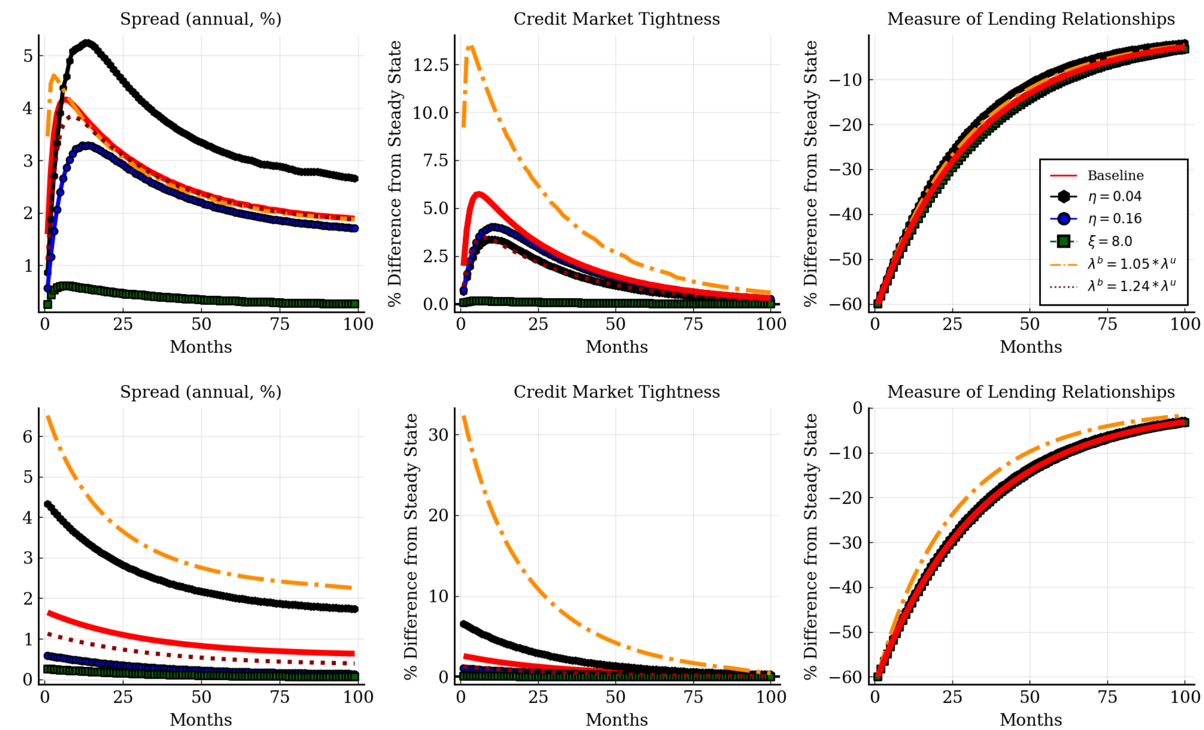

Figure 11: Robustness of optimal policy with commitment (top) and without commitment (bottom)

Figure 11 plots the optimal policy outcomes following a 60% destruction of relationships under commitment (top panels) and without commitment (bottom panels). The main takeaway is that the qualitative features of our baseline example are robust. The time path of the interest spread is hump-shaped under the Ramsey solution while it is downward-sloping in the absence of commitment. Further, the magnitude of the spreads tend to be higher with commitment, and the recovery of lending relationships is faster. In terms of differences, when lending relationships have a small effect on innovation or when banks have low bargaining power, the monetary authority sets larger spreads than under the baseline. Vice versa, when lending relationships have a lower impact on innovation or when banks have a higher bargaining power.

Specification	ρ	ξ	δ	$\bar{\alpha}$	λ^u	λ^b	B	η	ζ
Baseline	0.0033	15.7	0.01	0.46	0.029	0.034	3.42	0.08	0.015
$\eta = 0.04$	0.0033	15.7	0.01	0.93	0.029	0.034	3.42	0.04	0.015
$\eta = 0.16$	0.0033	15.7	0.01	0.23	0.029	0.034	3.42	0.16	0.015
$\lambda^b = 1.05\lambda^u$	0.0033	15.7	0.01	0.85	0.029	0.030	3.42	0.08	0.015
$\lambda^b = 1.28\lambda^u$	0.0033	15.7	0.01	0.34	0.029	0.037	3.42	0.08	0.015
$\xi = 8.0$	0.0033	8.0	0.01	0.48	0.029	0.034	3.42	0.08	0.015